Exercise sheet 6

1. Use Proposition 3.38 (3) to show that $\exp_{\mathrm{GL}(n,\mathbb{R})}(tA) = \mathrm{Exp}(tA)$ for all $t \in \mathbb{R}$ and $A \in \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}_{n,n}(\mathbb{R})$, where Exp denotes the matrix exponential.

2. Show that $\mathrm{Exp}\colon \mathfrak{u}(n) \to U(n)$ is surjective.

   Hint: Combine the fact that every $A \in U(n)$ is diagonalizable with the formula $g\mathrm{Exp}(X)g^{-1} = \mathrm{Exp}(gXg^{-1})$, which is valid for all $X \in \mathrm{Mat}_{n,n}(\mathbb{C})$ and $g \in \mathrm{GL}(n, \mathbb{C})$.

3. Show that $\mathrm{Exp}\colon \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is surjective.

   Hint: Use a similar argument as in Exercise 2 and the Jordan normal form.

4. Let V be a finite dimensional real vector space and $\Gamma < V$ a discrete subgroup. Show that there exist $\gamma_1, \ldots, \gamma_r \in \Gamma$, linearly independent in V such that $\Gamma = \mathbb{Z}\gamma_1 + \ldots + \mathbb{Z}\gamma_r$.

5. Show that every connected abelian Lie group G is isomorphic as Lie groups to $\mathbb{T}^a \times \mathbb{R}^{n-a}$ for some $a \in \{0, \ldots, n\}$, where $n = \dim G$ and $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.

6. Let G be a Lie group. Show that there is an open neighborhood of e which does not contain any non-trivial subgroup of G.