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# Exercise sheet 7

1. Review the proof of Cartan's Theorem.

2. Let $H < G$ be a closed subgroup of a Lie group $G$ with Lie algebra $\mathfrak{g}$. Show that

$$\mathrm{Lie}(H) = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \, \forall t \in \mathbb{R}\}.$$

3. Show that a continuous group homomorphism between two Lie groups is smooth.

   Hint: Look at the graph of the map and apply Cartan's theorem.

4. Read the pages 32-34 in *Representations of Compact Lie Groups* by Bröcker-Dieck.

5. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Show that $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}(g)$ is smooth, where $\mathrm{Ad}(g) := D_e(\mathrm{int}(g))$.

   Hint: Apply Proposition 3.50 to the map $\mathrm{int}(g)\colon G \to G$, $x \mapsto gxg^{-1}$. Use that $\exp_G$ is a local diffeomorphism to conclude that Ad is smooth near $e$. Then use left translation to show that Ad is smooth everywhere.

6. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \lhd \mathfrak{g}$ an abelian ideal in $\mathfrak{g}$. Show that $\exp_G(\mathfrak{a})$ is a normal subgroup of $G$.

7. Let $G$ be a topological group and $H < G$ a closed subgroup. Show that if $H$ and $G/H$ are connected, then so is $G$.