Exercise sheet 8

1. Convince yourself that all the definitions and results in Chapter 4 concerning uniquely real Lie algebras go over without modifications to complex Lie algebras.

2. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{g}$ as a vector space. Show that the bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ extends uniquely to a $\mathbb{C}$-bilinear map $[\cdot,\cdot]_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ turning $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra.

3. In the setting of Exercise 2 show that the canonical injection $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$, $X \mapsto X \otimes 1$ is a homomorphism of real Lie algebras and, if we identify $\mathfrak{g}$ with its image in $\mathfrak{g}_{\mathbb{C}}$, we have that
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}.$$
Express the bracket of $\mathfrak{g}_{\mathbb{C}}$ in this decomposition.

4. In the setting of Exercises 2 and 3 show that
   (a) $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{\mathbb{C}}$ is solvable,
   (b) $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}_{\mathbb{C}}$ is nilpotent.