# Exercise sheet 9

1. Read pages 4-37 to 4-39 and verify that everything works as stated.

2. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and Killing form $K_\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and $g \in G$
$$K_\mathfrak{g}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = K_\mathfrak{g}(X, Y),$$
where $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of $G$.

   Hint: Compute the derivative of $\Phi(t) := K_\mathfrak{g}(\mathrm{Ad}(\exp tZ)X, \mathrm{Ad}(\exp tZ)Y)$ and use Proposition 4.46.

3. Let $\mathfrak{g}$ be a real Lie algebra.
   (a) Show that under the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ we have that
   $$K_\mathfrak{g} = K_{\mathfrak{g}_\mathbb{C}|_{\mathfrak{g}\times\mathfrak{g}}}.$$
   (b) Show that $K_{\mathfrak{g}_\mathbb{C}}|_{\mathfrak{g}_\mathbb{C}^{(1)} \times \mathfrak{g}_\mathbb{C}^{(1)}} = 0$ if and only if $K_\mathfrak{g}|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}} = 0$.

   Hint: Use that $\mathfrak{g}_\mathbb{C}^{(1)} = \mathfrak{g}^{(1)} + i\mathfrak{g}^{(1)}$.

4. Compute the Killing form of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

5. Consider the three-dimensional Heisenberg group
$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$
Note that the center of $H$ is
$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$
Let $D < Z(H)$ be the following discrete subgroup
$$D := \mathrm{SL}_3(\mathbb{Z}) \cap Z(H) = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$
Check that $G := H/D$ is a connected, solvable Lie group and show that $G$ does not admit a smooth, injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space $V$ .

6. Show the following exceptional isomorphisms of Lie algebras.

   (a) Show that $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$. Hint: If $\dim V = 4$ then $\dim \Lambda^2 V = 6$.

   (b) Show that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

   (c) Show that $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$.

   (d) Show that $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$.

7. The goal of this exercise is to show that the spin group $\mathrm{SU}(2,\mathbb{C})$ is the universal covering group of the rotation group $\mathrm{SO}(3,\mathbb{R})$. Consider

$$\mathrm{SU}(2,\mathbb{C}) := \{g \in \mathrm{SL}(2,\mathbb{C}) : g^* g = I\} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \right\}$$

and its Lie algebra

$$\mathfrak{su}(2,\mathbb{C}) = \{X \in \mathfrak{sl}(2,\mathbb{C}) : X^* + X = 0\} = \left\{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} : a \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

   (a) Construct a Lie group homomorphism $\varphi : \mathrm{SU}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{R})$ whose kernel is $\{\pm I\}$.

   Hint: Use the adjoint representation of $\mathrm{SU}(2,\mathbb{C})$ on $\mathfrak{su}(2,\mathbb{C})$ and show that

$$b(X,Y) := -\frac{1}{2}\mathrm{tr}(XY)$$

   defines a positive definite symmetric bilinear form on $\mathfrak{su}(2,\mathbb{C})$ considered as a real vector space.

   (b) Show that $d_I\varphi : \mathfrak{su}(2,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{R})$ is a Lie algebra isomorphism and deduce that $\varphi$ is a covering map.

   (c) Show that $\mathrm{SU}(2,\mathbb{C})$ is homeomorphic to the 3-sphere $\mathbb{S}^3$ and deduce that $\mathrm{SU}(2,\mathbb{C})$ is simply connected. Show that $\mathrm{SO}(3,\mathbb{R})$ is homeomorphic to the three-dimensional real projective space $\mathbb{RP}^3$. What is the fundamental group of $\mathrm{SO}(3,\mathbb{R})$?

   (d) Are there any other Lie groups whose Lie algebra is isomorphic to $\mathfrak{su}(2,\mathbb{C})$?

   Hint: Analyze the discrete normal subgroups of $\mathrm{SU}(2,\mathbb{C})$.