at a point $p = (x_0, y_0)$ which is a simple extremum or saddle point of the function $f(x, y)$. The level curves $f(x, y) =$ constant are the dotted lines orthogonal to the integral curves. If p is a singularity of a general vector field, the pattern can be more complicated; possibilities are shown in (c) and (d). Interesting relations between the topological nature of the surface and the possible types of singularities possessed by a vector field on it were discovered by Poincaré, Hopf, and others (see Milnor [2]). A consequence of these relations is the fact already mentioned that a vector field on S^2—in fact on any closed orientable surface except T^2—must have at least one singular point.

Another important question about a vector field X on M is whether or not it has closed integral curves—diffeomorphic to the circle S^1 (see Exercise 3). This can be of importance, for instance, in applications to dynamics. In these applications one considers the points of a manifold as corresponding to, or parametrizing, the states of a dynamical system. For example, if the system consists of the earth, sun, and moon, then in a fixed coordinate system the positions of the three objects can be characterized by nine numbers (three sets of coordinates) and their velocities, or momenta, by nine more (the components of three vectors). Thus each state or configuration corresponds to a point on a manifold M of dimension 18. The laws of motion can be expressed as a system of ordinary differential equations or vector field X on M, and the integral curves correspond to the motions beginning from various initial states. A closed integral curve corresponds to a periodic motion, like that of the planets. This approach to mechanics was extensively studied by Poincaré and Birkhoff, and is still an active area of research (see Smale [2]). It has led to many interesting questions about vector fields and curves on manifolds. For example, it was very recently shown by Schweitzer [1], that there exist everywhere regular vector fields on S^3 without any closed integral curves—contrary to a long standing conjecture. Classical mechanics in the framework of manifold theory is very clearly set forth by Godbillon [1]. An excellent recent book on differential equations and dynamical systems is Hirsch and Smale [1].

(5.5) Definition A vector field X on M is said to be *complete* if it generates a (global) action of R on M, that is, if $W = R \times M$.

This is clearly the most desirable case and we find it very convenient to have sufficient conditions for completeness. One of them is an immediate corollary of Lemma 5.1.

(5.6) Corollary *If M is a compact manifold, then every vector field X on M is complete.*

To see that this is so we take $K = M$ in the lemma and note that in this case $\alpha(p) = -\infty$ and $\beta(p) = +\infty$, that is, $I(p) = R$, for every $p \in M$.

This gives one important case in which we may be sure that a vector field is complete. A second case, which we will study in some detail, is a left-invariant vector field on a Lie group, as is shown by the corollary to the theorem which follows.

(5.7) Theorem *Let X be a C^∞-vector field on a manifold M and $F\colon M \to M$ a diffeomorphism. Let $\theta(t, p)$ denote the C^∞ map $\theta\colon W \to M$ defined by X. Then X is invariant under F if and only if $F(\theta(t, p)) = \theta(t, F(p))$ whenever both sides are defined.*

Proof Suppose that X is invariant under F. If $\theta_p\colon I(p) \to M$ is the integral curve of X with $\theta_p(0) = p$, then the diffeomorphism F takes it to an integral curve $F(\theta_p(t))$ of the vector field $F_*(X)$. Since $F_*(X) = X$ and $F(\theta_p(0)) = F(p)$, from uniqueness of integral curves we conclude that $F(\theta_p(t)) = \theta(t, F(p))$. This proves the "only if" part of the theorem.

Now suppose that $F(\theta(t, p)) = \theta(t, F(p))$ and prove that $F_*(X_p) = X_{F(p)}$. This could be done directly from expression (3.2) for the infinitesimal generator X, but we shall proceed in a slightly different way. Let $\theta_p(t) = \theta(t, p)$ and let d/dt be the natural basis of $T_0(R)$, the tangent space to R at $t = 0$. Then, by definition, $X_p = \dot{\theta}_p(0) = \theta_{p*}(d/dt)$ and applying the isomorphism $F_*\colon T_p(M) \to T_{F(p)}(M)$ to this definition we have

$$F_*(X_p) = F_* \circ \theta_{p*}(d/dt) = (F \circ \theta_p)_*(d/dt) = \theta_{F(p)*}(d/dt) = X_{F(p)}.$$

The second equality is the chain rule for the composition of mappings applied to $\theta_p\colon R \to M$ and $F\colon M \to M$. The third equality uses the hypothesis that $F \circ \theta_p(t) = \theta_{F(p)}(t)$. ∎

We remark that in the notation of Section 3 this theorem could be stated: $F_*(X) = X$ *if and only if* $\theta_t \circ F = F \circ \theta_t$ on V_t.

(5.8) Corollary *A left-invariant vector field on a Lie group G is complete.*

Proof Let X be such a vector field. There is a neighborhood V of e and a $\delta > 0$ such that $\theta(t, g)$ is defined on $I_\delta \times V$. For $h \in G$, let L_h denote the left translation by h. If we apply Theorem 5.7 with $F = L_h$, then $\theta(t, L_h g) = L_h \theta(t, g)$, which shows that θ is defined on $I_\delta \times L_h(V)$, a neighborhood of $(0, h)$ in $R \times G$. It follows that for every $h \in G$ there is a neighborhood $U = L_h(V)$ such that $I_\delta \times U \subset W$, the domain of θ with the same $\delta > 0$ that we first obtained for V, that is, δ is fixed and independent of h. By the same argument as in the compact case we obtain a contradiction if we assume for any $g \in M$ that either $\alpha(g)$ or $\beta(g)$ is finite. Therefore $W = R \times M$ and X is complete. ∎

We now make use of a vector field X on M to define a method of differentiation which has many applications in manifold theory. We have already defined the derivative of a function $f \in C^\infty(M)$ at a point p in the direction of X; it is just $X_p f$. This generalizes from $\boldsymbol{R}^n$ to an arbitrary manifold the notion of directional derivative of a function. However, if we wish to determine the rate of change of a vector field Y at $p \in M$ in some direction X_p, we have trouble as soon as we leave $\boldsymbol{R}^n$, for it is only in $\boldsymbol{R}^n$ that we are able to compare the value of Y at p with its value at nearby points, which we must do to compute a rate of change. Now, given a vector field X on M, there is an associated one-parameter group $\theta: W \to M$ generated by X. For each $t \in R$ we know (Theorem 3.12) that $\theta_t: V_t \to V_{-t}$ is a diffeomorphism (with inverse θ_{-t}) of the open set V_t, provided V_t is not empty. In particular for each $p \in M$ there is a neighborhood V and a $\delta > 0$ such that $V \subset V_t$ for $|t| < \delta$. The isomorphism $\theta_{t*}: T_p(M) \to T_{\theta_t(p)}(M)$ and its inverse allow us to compare the values of vector fields at these two points.

Indeed, suppose Y is a second C^∞-vector field on M. We may use this idea to compute for each p the rate of change of Y in the direction of X, that is, along the integral curve of the vector field X passing through p. We shall denote this rate of change by $L_X Y$; it is itself a C^∞-vector field.

(7.6) Definition The vector field $L_X Y$, called the *Lie derivative* of X with respect to Y is defined at each $p \in M$ by either of the following limits.

$$\begin{aligned}(L_X Y)_p &= \lim_{t \to 0} \frac{1}{t}\left[\theta_{-t*}(Y_{\theta(t,p)}) - Y_p\right] \\ &= \lim_{t \to 0} \frac{1}{t}\left[Y_p - \theta_{t*} Y_{\theta(-t,p)}\right].\end{aligned}$$

The second definition is obtained from the first by replacing t by $-t$. We interpret the first expression as follows: Apply to $Y_{\theta(t,p)} \in T_{\theta(t,p)}(M)$ the isomorphism θ_{-t*}, taking $T_{\theta(t,p)}(M)$ to $T_p(M)$. Then in $T_p(M)$ take the difference of this vector and Y_p, multiply by the scalar $1/t$, and pass to the limit as $t \to 0$. This limit is a vector $(L_X Y)_p \in T_p(M)$; if it exists at all, that is! The existence as well as the fact that the vector field so defined is C^∞ may be verified by writing the formula above in local coordinates (Exercise 6). We shall give another characterization of $L_X Y$ which requires a modification of Lemma II.4.3, following Kobayashi and Nomizu [1, p. 15].

(7.7) Lemma *Let X be a C^∞-vector field on M and θ be the corresponding map of $W \subset R \times M$ onto M. Given $p \in M$ and $f \in C^\infty(U)$, U an open set containing p, we choose $\delta > 0$ and a neighborhood V of p in U such that*

$\theta(I_\delta \times V) \subset U$. Then there is a C^∞ function $g(q, t)$ defined on $V \times I_\delta$ such that for $q \in V$ and $t \in I_\delta$ we have

$$f(\theta_t(q)) = f(q) + tg(q, t) \qquad \textit{and} \qquad X_q f = g(q, 0).$$

Proof There is a neighborhood V of p and a $\delta > 0$ such that $\theta_t(p) = \theta(t, p)$ is defined and C^∞ on $I_\delta \times V$ and maps $I_\delta \times V$ into U according to Theorem 4.2. The function $r(t, q) = f(\theta_t(q)) - f(q)$ is C^∞ on $I_\delta \times V$ and $r(0, q) = 0$. We denote by $\dot{r}(t, q)$ its derivative with respect to t. We define $g(q, t)$—for each fixed q—by the formula

$$g(q, t) = \int_0^1 \dot{r}(ts, q)\, ds.$$

This function is also C^∞ on $I_\delta \times V$ (verified by use of local coordinates and properties of the integral). By the fundamental theorem of calculus,

$$tg(q, t) = \int_0^1 \dot{r}(ts, q)t\, ds = r(t, q) - r(0, q) = r(t, q).$$

Using the definition of r, this becomes

$$f(\theta_t(q)) = f(q) + tg(q, t).$$

On the other hand, by the definition (3.2) of the infinitesimal generator of θ,

$$g(q, 0) = \lim_{t \to 0} g(q, t) = \lim_{t \to 0} \frac{1}{t} r(t, q) = \lim_{t \to 0} \frac{1}{t} [f(\theta_t(q)) - f(q)] = X_q f. \quad ∎$$

We use the lemma to prove the following theorem:

(7.8) Theorem *If X and Y are C^∞-vector fields on M, then*

$$L_X Y = [X, Y].$$

Proof By definition

$$(L_X Y)_p f = \left(\lim_{t \to 0} \frac{1}{t} [Y_p - \theta_{t*}(Y_{\theta_{-t}(p)})]\right) f.$$

This differential quotient and that of the following expression, whose limit is the derivative of a C^∞ function of t, are equal for all $0 < |t| < \delta$; hence equal in the limit

$$(L_X Y)_p f = \lim_{t \to 0} \frac{1}{t} [Y_p f - Y_{\theta_{-t}(p)}(f \circ \theta_t)].$$

Using Lemma 7.7 and denoting $g(q, t)$ by g_t, we have

$$(L_X Y)_p f = \lim_{t \to 0} \frac{1}{t} [Y_p f - Y_{\theta_{-t}(p)}(f + tg_t)].$$

Then replace t by $-t$ and rearrange terms giving

$$(L_X Y)_p = \lim_{t \to 0} \frac{1}{t} [(Yf)(\theta_t(p)) - (Yf)(p)] - \lim_{t \to 0} Y_{\theta_t(p)} g_t .$$

Now, using both the formula (3.2) with f replaced by Yf and Δt by t, and the fact that $g_0 = g(q, 0) = Xf(q)$, we obtain in the limit

$$(L_X Y)_p f = X_p(Yf) - Y_p(Xf) = [X, Y]_p f.$$

This completes the proof of the theorem; it also shows that $L_X Y$ is C^∞. ∎

(7.9) Theorem *Let $F: N \to M$ be a C^∞ mapping and suppose that X_1, X_2 and Y_1, Y_2 are vector fields on N, M, respectively, which are F-related, that is, for $i = 1, 2$, $F_*(X_i) = Y_i$. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related, that is, $F_*[X_1, X_2] = [F_*(X_1), F_*(X_2)]$.*

Proof Before proving the theorem we note the following necessary and sufficient condition for X on N and Y on M to be F-related: for any g which is C^∞ on some open set $V \subset M$,

$$(*) \qquad (Y_g) \circ F = X(g \circ F)$$

on $F^{-1}(V)$. This is essentially a restatement of the definition of F-related, for if $q \in F^{-1}(V)$, then $F_*(X_q)g = X_q(g \circ F) = X(g \circ F)(q)$; and $Y_{F(q)}g$ is the value of the C^∞ function Yg at $F(q)$, that is, $((Yg) \circ F)(q)$. Thus the condition holds if and only if $F_*(X_q) = Y_{F(q)}$ for all $q \in M$.

Returning to the proof we consider $f \in C^\infty(V)$, $V \subset M$, so that $Y_1 f$ and $Y_2 f \in C^\infty(V)$ also. Apply $(*)$, first with $g = Y_2 f$ and then with $g = f$ giving the equalities

$$[Y_1(Y_2 f)] \circ F = X_1((Y_2 f) \circ F) = X_1[X_2(f \circ F)].$$

Interchanging the roles of Y_1, Y_2 and X_1, X_2 and subtracting, we obtain

$$([Y_1, Y_2]f) \circ F = [X_1, X_2](f \circ F),$$

which according to $(*)$ is equivalent to $[X_1, X_2]$ and $[Y_1, Y_2]$ being F-related. ∎

We now define the Lie algebra $\mathfrak{g}$ of a Lie group G.

(7.10) Corollary *If G is a Lie group, then the left-invariant vector fields on G form a Lie algebra $\mathfrak{g}$ with the product $[X, Y]$ and* $\dim \mathfrak{g} = \dim G$. *If $F: G_1 \to G_2$ is a homomorphism of Lie groups, $F_*: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras.*

Proof Let $a \in G$, and let X and Y be left-invariant vector fields. L_a (left translation) is a diffeomorphism and $L_{a*}X = X$, $L_{a*}Y = Y$. Therefore $L_{a*}[X, Y] = [X, Y]$ by the theorem, so $[X, Y]$ is L_a-invariant for any a. Hence the subspace $\mathfrak{g}$ of left-invariant vector fields is closed with respect to $[X, Y]$. Since each $X \in \mathfrak{g}$ is uniquely determined by X_e, the mapping $X \to X_e$ is an isomorphism of $\mathfrak{g}$ and $T_e(G)$ as vector spaces. The last statement follows from Corollary 2.10 and Theorem 7.9. ∎

(7.11) Remark If $H \subset G$ is a Lie subgroup, then Corollary 7.10 implies that $i_*(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$. It consists of the elements of $\mathfrak{g}$ tangent to H and its cosets gH.

(7.12) Theorem *Let X and Y be complete C^∞-vector fields on a manifold M and let θ, σ denote the corresponding actions of R on M. Then $\theta_t \circ \sigma_s = \sigma_s \circ \theta_t$ for all $s, t \in R$ if and only if $[X, Y] = 0$.*

Proof We first suppose that $\theta_t \circ \sigma_s = \sigma_s \circ \theta_t$ for all $s, t \in R$. Applying Theorem 5.7 to the diffeomorphism $\theta_t: M \to M$, we see that Y is θ_t-invariant; in particular $\theta_{t*} Y = Y$. This implies that

$$[X, Y] = L_X Y = \lim_{\Delta t \to 0} [Y - \theta_{-t*} Y] = 0.$$

Next assume $[X, Y] = 0$, then from the previous theorem

$$0 = \theta_{t*}[X, Y] = [\theta_{t*} X, \theta_{t*} Y] = [X, \theta_{t*} Y].$$

From this we conclude that for any $p \in M$ and any $f \in C^\infty(p)$ we have

$$0 = (L_X(\theta_{t*} Y))_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [(\theta_{t*} Y)_p f - (\theta_{t-\Delta t*} Y)_p f]$$

so that $d(\theta_{t*} Y)_p f/dt = 0$ for every t, that is, $(\theta_{t*} Y)_p f$ is constant. When $t = 0$ this constant function has the value $Y_p f$, therefore $(\theta_{t*} Y)_p f = Y_p f$. Since p and $f \in C^\infty(p)$ were arbitrary, it follows that $\theta_{t*} Y = Y$ and from Theorem 5.7 we conclude that for each $t \in R$

$$\theta_t \circ \sigma_s = \sigma_s \circ \theta_t .$$ ∎

Exercises

1. Show that $\mathfrak{X}(M)$ is infinite-dimensional over R but locally finitely generated over $C^\infty(M)$, that is, each $p \in M$ has a neighborhood V on which there is a finite set of vector fields which generate $\mathfrak{X}(M)$ as a $C^\infty(V)$ module.
2. Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ and prove that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$