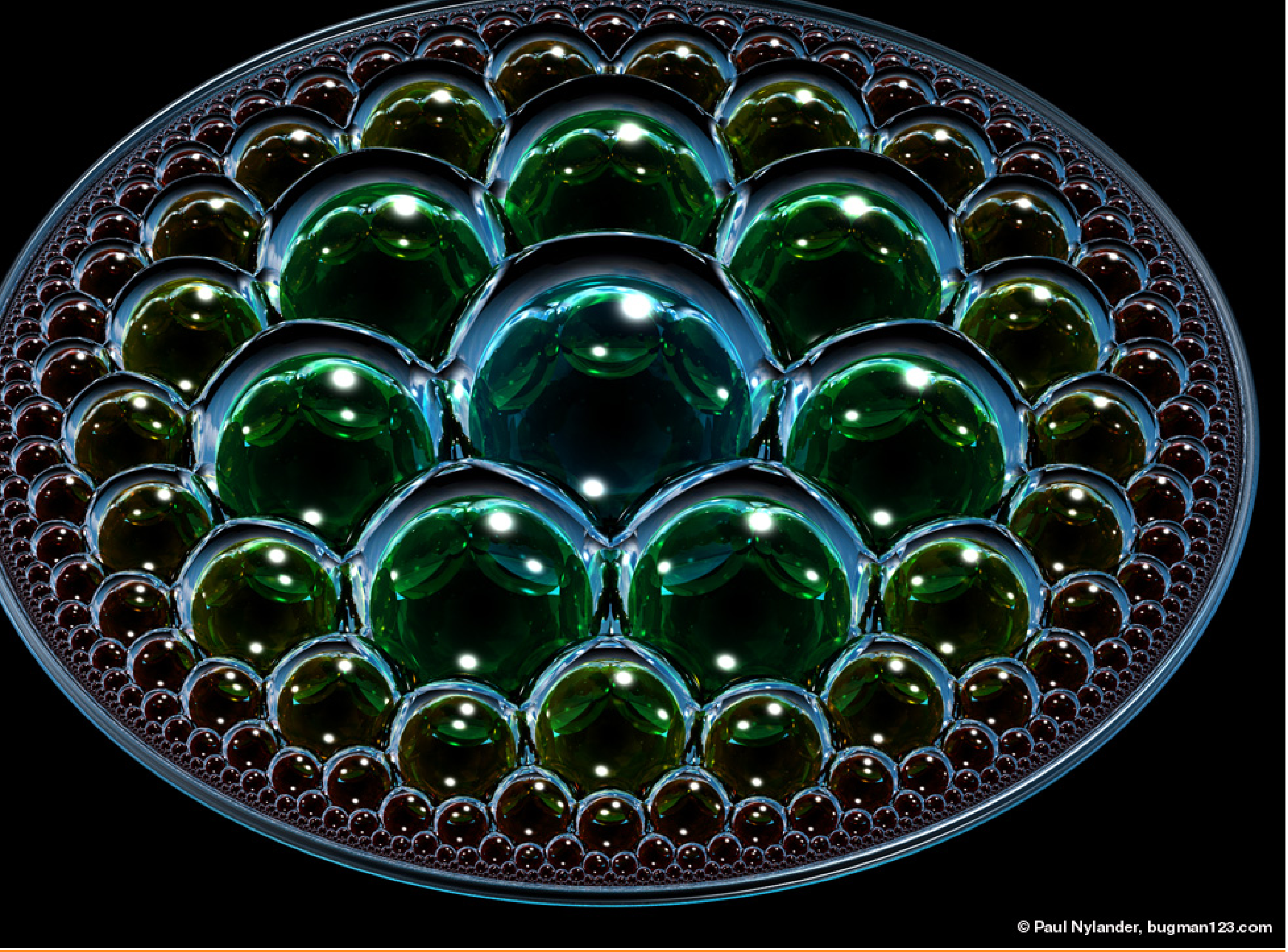

From Topological Groups to Lie Groups

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Chapter 1 Introduction

Lie groups are named after Sophus Lie, a Norwegian mathematician of the second half of the nineteenth century who developed the theory of continuous transformation groups. His original idea was to develop a theory of symmetries of differential equations parallel to the theory developed by Galois for algebraic equations, with Lie groups being the continous analogue of permutation groups in Galois theory. This point of view did not fulfill Lie's expectations and went in unexpected directions (see for example the theory of differential fields, D-modules, etc), but Lie groups came to be an indispensable tool in many branches of mathematics as well as in theoretical physics.

The definition of a Lie group is simple, in that it is a differentiable manifold that is also a group, such that the group operations are compatible with the manifold structure. One can hence study Lie groups from a geometrical point of view or from an algebraic point of view. The starting point of the algebraic point of view is the existence of an algebraic object, namely the Lie algebra of the Lie group, that turns out to have the geometric interpretation as the tangent space at the identity of the Lie group. There is also a somewhat different approach, much more elementary, but also more restrictive, in which one considers only linear Lie groups, that is (closed) subgroups of $\mathrm{GL}(n,\mathbb{R})$ and one develops the whole theory via elementary methods. While this is appropriate for example in case one is teaching a course to students with limited mathematical background, there is the problem that, although any Lie group can be locally realized as a linear Lie group, there are important Lie groups that are not (globally) linear. Our approach will be the more algebraic one. In fact, as such we will start by considering Lie groups just as topological groups, that is topological spaces that are also groups, such that the group operations are compatible with the topological structure. We will see how far one can go for topological groups, and we will see that there are some miraculous facts that arise from the interplay of these two structures.

Chapter 2 Topological Groups

1. Check Exercise numbering

2.1 Definitions and Examples

Definition 2.1. Topological group

A topological group G *is a group endowed with a topology with respect to which the group operations*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are continuous. ♣

Remark The following “regularity" properties follows simply from the definition:

- The inversion $g \mapsto g^{-1}$ is a continuous bijection. Since its inverse $g^{-1} \mapsto (g^{-1})^{-1}$ is also continuous, then it is a homeomorphism.
- The left translation

$$\begin{aligned} L_g \colon G &\to G \\ x &\mapsto gx \end{aligned}$$

  and the right translation

$$\begin{aligned} R_g \colon G &\to G \\ x &\mapsto xg \end{aligned}$$

  are continuous and bijective. Since $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$ are also continuous, L_g and R_g are homeomorphisms. If $U \ni e$ is a neighborhood of the identity (that is a set in G containing e and and open set $U_e \ni e$), then L_gU is a neighborhood of g homeomorphic to U. Hence topological groups "look everywhere the same".

- If G_1, G_2 are topological groups and $\rho\colon G_1 \to G_2$ is a homomorphism, then ρ is continuous if and only if it is continuous at one point.

Remark. Before we proceed to give concrete examples of topological groups, we remark that there are simple operations that preserve the class of topological groups.

- Any subgroup of a topological group is a topological group (see also Proposition 2.1.3.).
- Products of topological groups are topological groups.
- Quotients of topological groups are also topological groups
- The semidirect product of topological groups is a topological group. We recall in fact that if H, N are topological groups and $\eta\colon H \to \mathrm{Aut}(N)$ is a homomorphism such that
$$\begin{aligned} H \times N &\to N \\ (h,n) &\mapsto \eta(h)n \end{aligned}$$
is continuous, the semidirect product $H \ltimes_\eta N$ is the Cartesian product $H \times N$ with the product
$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 \eta(h_1) n_2)$$
for all $h_1, h_2 \in H$ and $n_1, n_2 \in N$, and it is a topological group with the product topology. Notice that there are other characterizations of a semidirect product. We recall these here since we will be using it in the sequel.

Lemma 2.1

Let G be a topological group, $H < G$ a closed subgroup and $N \trianglelefteq G$ a closed normal subgroup. The following are equivalent:

1. *There exists a homomorphism $\eta\colon H \to \mathrm{Aut}(N)$ such that $G = H \ltimes_\eta N$;*
2. *G is a group extension of N by H, that is there exists a short exact sequence*
$$\{e\} \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow \{e\}\,.$$
that splits, that is the composition $p \circ i\colon H \to G/N$ of the embedding $i\colon H \hookrightarrow G$ and the natural projection $p\colon G \to G/N$ is an isomorphism of topological groups.

♡

Example 2.1 Any group with the discrete topology is a topological group. In this case any subset is open and any map to any other topological group is continuous.

Example 2.2 The vector space $(\mathbb{R}^n, +)$ with the componentwise addition is a commutative topological group in the Euclidean topology.

Example 2.3 The non-zero real numbers and the non-zero complex numbers, $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$, are commutative topological groups with the topology induced by the Euclidean topology.

Example 2.4 Let us denote by $\mathbb{R}^{n\times n}$ the vector space of $n \times n$ matrices with real coefficients and let us define

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\} .$$

Then $\mathrm{GL}(n, \mathbb{R})$ is an open set in $\mathbb{R}^{n\times n}$ and it inherits from $\mathbb{R}^{n\times n}$ the Euclidean topology. With this topology $\mathrm{GL}(n, \mathbb{R})$ is a topological group. In fact the topology on $\mathbb{R}^{n\times n}$, and hence on $\mathrm{GL}(n, \mathbb{R})$ is such that if $(A_k)_{k\in\mathbb{N}} \subset \mathrm{GL}(n, \mathbb{R})$ is a sequence, then

$$A_k \to A \text{ if and only if } (A_k)_{ij} \to A_{ij}$$

for all $1 \le i, j \le n$. Since if $A, B \in \mathrm{GL}(n, \mathbb{R})$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} ,$$

this means that the multiplication is continuous. Since

$$(A^{-1})_{ij} = \frac{\det M_{ji}}{\det A} ,$$

where M_{ji} is the (j, i)-minor matrix obtained by removing the i-th row and the j-th column and by multiplying by $(-1)^{i+j}$, then the inversion is also continuous.

Example 2.5 In the Example 2.4 we used that $\mathbb{R}$ is a *topological field*, that is the sum, the multiplication and the inversion are continuous. As a consequence, on $\mathbb{R}^{n\times n}$ there is a topology that induces the topology on $\mathrm{GL}(n, \mathbb{R})$. Likewise, if $\mathbb{F}$ is any topological field, then $\mathrm{GL}(n, \mathbb{F})$ is a topological group. Examples of topological fields are $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ and finite fields. Here $\mathbb{Q}_p$ is the field of p-adic integers, which can be defined as the field of fractions of the ring of p-adic integers $\mathbb{Z}_p$ defined in Example 2.10.

Example 2.6 Let X be a compact Hausdorff space. Then

$$\mathrm{Homeo}(X) := \{f : X \to X : f \text{ is a homeomorphism}\}$$

is a topological group with the compact-open topology (see Definition A.5).

If X is only locally compact but not compact, then $\mathrm{Homeo}(X)$ need not be a topological group. If however X is locally compact but also locally connected, then $\mathrm{Homeo}(X)$ is a topological group. This includes for example all manifolds. Likewise, if X is a proper metric space (that is a metric space in which closed balls of finite radius are compact), then $\operatorname{Homeo} X$ is a topological group.

Example 2.7 Let (X, d) be a compact metric space and let

$$\operatorname{Iso}(X) := \{f \in \operatorname{Homeo}(X) : d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X\}.$$

Then $\operatorname{Iso}(X) \subset \operatorname{Homeo}(X)$ is a closed subgroup and hence a topological group (Exercise 1.).

Example 2.8 We showed in Example 2.4 that $\mathrm{GL}(n, \mathbb{R})$ is a topological group when it inherits the Euclidean topology as a subspace of $\mathbb{R}^{n\times n}$. We show now that $\mathrm{GL}(n, \mathbb{R})$ is a topological group also with respect to the compact-open topology (and in fact the two topologies coincide, see Exercise 2.). In fact, since $\mathrm{GL}(n, \mathbb{R}) < \operatorname{Homeo}(\mathbb{R}^n)$ and $\mathbb{R}^n$ is a proper metric space, then the compact-open topology on $\mathrm{GL}(n, \mathbb{R}) < \operatorname{Homeo}(\mathbb{R}^n)$ is the topology of the uniform convergence on compact sets. If $(A_k)_{k\in\mathbb{N}} \subset \mathrm{GL}(n, \mathbb{R})$, and $A_k \to A$ uniformly on compact set, then A is linear, so that $\mathrm{GL}(n, \mathbb{R})$ is a (closed) subgroup of $\operatorname{Homeo}(\mathbb{R}^n)$ and is hence a topological group. Notice that for the limit of a sequence of linear functions to be linear, it is actually enough that the sequence converges pointwise.

Example 2.9 Let M be a smooth manifold. Then

$$\operatorname{Diff}^r(M) := \{f \in \operatorname{Homeo}(M) : f, f^{-1} \text{ are continuous and differentiable } r \text{ times}\}$$

is a subgroup of $\operatorname{Homeo}(M)$, hence a topological group, which however is not closed in the compact-open topology. We can consider however the C^r-topology, that is the topology according to which $(f_n)_{n\in\mathbb{N}} \xrightarrow{C^r} f$ if in any local chart $\psi\colon U \to \mathbb{R}^n$, $U \subset M$, the sequence $(f_n \circ \psi^{-1})_{n\geq 1}$ and all its partial derivatives up to order r converge uniformly on compact sets to the corresponding derivatives of $f \circ \psi^{-1}$. With this topology $\operatorname{Diff}^r(M)$ is a topological group that is complete in a natural sense.

Example 2.10 Let Λ be a partially ordered set and let $(G_\lambda)_{\lambda\in\Lambda}$ be a family of groups such that for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \leq \lambda_2$ there exists a homomorphism

$$G_{\lambda_2} \xrightarrow{\rho_{\lambda_2,\lambda_1}} G_{\lambda_1}$$

satisfying the following properties:

1. $\rho_{\lambda,\lambda} = \mathrm{Id}|_{G_\lambda}$;
2. $\rho_{\lambda_3,\lambda_1} = \rho_{\lambda_2,\lambda_1} \circ \rho_{\lambda_3,\lambda_2}$ for all $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

Then the *inverse limit* G of the *projective system* $((G_\lambda)_{\lambda\in\Lambda}, \rho_{\lambda_2,\lambda_1})$ is defined as the unique smallest topological group G such that for all $\lambda \in \Lambda$ there exists a continuous homomorphism $\rho_\lambda\colon G \to G_\lambda$

with the property that the diagram

$$\begin{tikzcd} G \arrow[rr, "\rho_{\lambda_2}"] \arrow[dr, "\rho_{\lambda_1}"'] & & G_{\lambda_2} \arrow[dl, "\rho_{\lambda_2,\lambda_1}"] \\ & G_{\lambda_1} & \end{tikzcd} \tag{2.1}$$

commutes, $\rho_{\lambda_1} = \rho_{\lambda_2,\lambda_1} \circ \rho_{\lambda_2}$. One can verify that G can be identified with

$$\varprojlim G_\lambda := \left\{ (x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} G_\lambda : \rho_{\lambda_2,\lambda_1}(x_{\lambda_2}) = x_{\lambda_1} \right\}.$$

Points in $\varprojlim G_\lambda$ are said to be *compatible*. If the $(G_\lambda)_{\lambda\in\Lambda}$ are topological groups, so is $\prod_{\lambda\in\Lambda} G_\lambda$ with the product topology and, since $\varprojlim G_\lambda$ is a (closed) subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$, it is a topological group as well with the induced topology.

Of course if the $(G_\lambda)_{\lambda\in\Lambda}$ are compact then by Tychonoff theorem also $\varprojlim G_\lambda$ is compact. Moreover, if the $(G_\lambda)_{\lambda\in\Lambda}$ are discrete, then $\varprojlim G_\lambda$ is totally disconnected, that is the connected sets are the points. In fact, let $C \subset G$ be a connected set. Since $\rho_\lambda \colon G \to G_\lambda$ is continuous, then $\rho_\lambda(C)$ is connected and hence a point, say $x_\lambda \in G_\lambda$. By the commutativity of the diagram (2.1) the sequence $(x_\lambda)_{\lambda\in\Lambda}$ must be compatible and unique, so that C is the singleton $\{(x_\lambda)_{\lambda\in\Lambda}\}$.

If the groups in the projective system $(G_\lambda)_{\lambda\in\Lambda}$ are finite, the resulting inverse limit is called *profinite*. It follows from the previous observation that profinite groups are compact and totally disconnected. An important example is the group of *p-adic integers* $\mathbb{Z}_p$, which is a profinite group under addition. In fact $\mathbb{Z}_p$ is the inverse limit of the projective system

$$((\mathbb{Z}/p^n\mathbb{Z}), (\rho_{n,m} \colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z})_{n\geq m}),$$

where $\rho_{n,m}$ is the natural reduction $\mod p^m$ homomorphism. One can check that the topology on $\mathbb{Z}_p$ is the same as the topology arising from the p-adic valuation on $\mathbb{Z}_p$ and with this topology $\mathbb{Z}_p$ is a topological ring. By the characteristic property of $\mathbb{Z}_p$ there are maps $\rho_n \colon \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ which hare continuous ring homomorphisms. The kernel of ρ_n is the ideal $p^n\mathbb{Z}_p$ which is open since $\mathbb{Z}/p^n\mathbb{Z}$ is discrete. Since $\bigcap_{n\geq 1} p^n\mathbb{Z}_p = \{0\}$), the sequence $\{p^n\mathbb{Z}_p : n \geq p\}$ is a fundamental system of neighborhoods of 0. One shows then:

1. $x \in \mathbb{Z}_p$ is invertible if and only if $xz \notin \mathbb{Z}_p$;
2. if $U = \mathbb{Z}_p^\times$ is the group of invertible elements, then every $x \in \mathbb{Z}_p \setminus \{0\}$ can be written uniquely as $x = p^n u$, with $n \geq 0$ and $u \in U$.

With thia at hand, one shows that $\mathbb{Z}_p$ is an integral domain; its field of fractions is the field $\mathbb{Q}_P$ of

p-adic numbers and equals $\mathbb{Z}_p\left[\frac{1}{p}\right]$. It is a locally compact non-discrete Hausdorff field. In fact any such field of characteristic zero is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$.

Example 2.11 We consider now three subgroups of $\mathrm{GL}(n,\mathbb{R})$ that will turn out to play an extremely important role.

1. Let
$$A_{\det} := \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) : \lambda_i \neq 0, \text{ for } 1 \leq i \leq n \right\}.$$
Then $A_{\det}$ is an Abelian topological group as it is homomorphic and homeomorphic to $(\mathbb{R}^*)^n$.
2. Let
$$N := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \right\}.$$
be the group of upper triangular matrices with all 1s on the diagonal. Then N is a (closed) subgroup of $\mathrm{GL}(n,\mathbb{R})$ and is hence a topological group. However, in this case N is homeomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$ as a topological space, but not as a group, as for example N is not Abelian, unless $n \leq 2$.
3. Let
$$\begin{aligned} K := \mathrm{O}(\mathbb{R}^n, \langle\cdot\,,\,\cdot\rangle) &= \{X \in \mathrm{GL}(n,\mathbb{R}) : \langle Xv, Xw\rangle = \langle v, w\rangle \text{ for all } v, w \in \mathbb{R}^n\} \\ &= \{X \in \mathrm{GL}(n,\mathbb{R}) : \|Xv\| = \|v\| \text{ for all } v \in \mathbb{R}^n\} \\ &= \left\{X \in \mathrm{GL}(n,\mathbb{R}) : {}^tXX = \mathrm{Id}_n\right\} \end{aligned}$$
be the orthogonal group of the usual Euclidean inner product $\langle\cdot\,,\,\cdot\rangle$ or of the usual Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$. This is a topological group as it is a (closed) subgroup of $\mathrm{GL}(n,\mathbb{R})$. The standard notation for this group is
$$\mathrm{O}(n,\mathbb{R}) := \mathrm{O}(\mathbb{R}^n, \langle\cdot\,,\,\cdot\rangle).$$

Example 2.12 We may also consider inner products on a vector space with respect to which vectors might have negative length. Let V be a real vector space and let $B : V \times V \to \mathbb{R}$ be a non-degenerate symmetric bilinear form on V, that is:

1. (Non-degeneracy) Given $x \in V$ there exists $y \in V$ such that $B(x,y) \neq 0$;
2. (Symmetry) $B(v,w) = B(w,v)$ for all $v, w \in V$, and

3. (Bilinearity) $B(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 B(v_1, w) + \alpha_2 B(v_2, w)$ for all $v_1, v_1, w \in V$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$.

Incidentally, given such a non-degenerate symmetric bilinear form is equivalent to choosing an isomorphism $\lambda : V \to V^*$ of V with its dual V^* given by $B(v, w) = \lambda(v)w$. If Q is the quadratic form associated to B, $Q(v) := B(v, v)$, the *orthogonal group of* (V, B) or of (V, Q) is defined as

$$\begin{aligned}\mathrm{O}(V, B) &= \{A \in \mathrm{GL}(V) :\ B(Av, Aw) = B(v, w),\ \text{for all } v, w \in V\} \\ &= \{A \in \mathrm{GL}(V) :\ Q(Av) = Q(v),\ \text{for all } v \in V\}\,.\end{aligned}$$

This is a topological group as it is a closed subgroup of $\mathrm{GL}(V)$.

Recall that one can always choose a basis of V so that B can be written as

$$B_p(v, w) = -\sum_{j=1}^{p} v_j w_j + \sum_{j=p+1}^{n} v_j w_j \tag{2.2}$$

for some fixed p. Then B is positive definite if and only if $p = 0$. If $V = \mathbb{R}^n$ and B_p is as in (2.2), then it is customary to use the notation

$$\mathrm{O}(p, q) := \mathrm{O}(V, B_p)\,.$$

Notice that in the above discussion it is essential that V is a real vector space, since instead over the complex numbers all $\mathrm{O}(V, B)$ are isomorphic once the dimension of V is fixed. In fact we can perform a change of basis

$$(e_1, \dots, e_p, e_{p+1}, \dots, e_n) \mapsto (\imath e_1, \dots, \imath e_p, e_{p+1}, \dots, e_n)$$

so that in the new basis the bilinear form reads

$$B(v, w) = \sum_{j=1}^{n} v_j w_j\,. \tag{2.3}$$

The *orthogonal group of the symmetric bilinear form in* (2.3) is denoted by

$$\mathrm{O}(n, \mathbb{C}) = \mathrm{O}(V, B)\,,$$

where now V is a complex n-dimensional vector space.

Example 2.13 Let V be a complex vector space and $h\colon V \times V \to \mathbb{C}$ a Hermitian inner product, that is a positive definite Hermitian complex valued form that is linear in the first variable and antilinear in the second. The *unitary group* $U(V, h)$ is defined as

$$\begin{aligned}\mathrm{U}(V, h) :=& \{X \in \mathrm{GL}(V) :\ h(Xv, Xw) = h(v, w) \text{ for all } v, w \in V\} \\ =& \{X \in \mathrm{GL}(V) :\ X^* = X^{-1}\}\,,\end{aligned}$$

where X^* denotes the adjoint with respect to h. Notice that if $X \in \mathrm{U}(V, h)$, then $|\det X| = 1$. If

$h\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is the standard Hermitian inner product

$$h(x,y) := \sum_{j=1}^{n} x_j \overline{y}_j \,,$$

then we use the notation

$$\mathrm{U}(n) := \mathrm{U}(\mathbb{C}^n, h)\,.$$

Example 2.14 Let k be a topological field. The *special linear group* defined as

$$\mathrm{SL}(n,k) := \{X \in \mathrm{GL}(n,k) : \det X = 1\}$$

is a topological group as a subgroup of $\mathrm{GL}(n,k)$. We can thus define subgroups of all of the above linear groups by taking the intersection with $\mathrm{SL}(n,k)$ with the appropriate field. So for example

$$\begin{aligned}
\mathrm{SO}(n,\mathbb{R}) &:= \mathrm{SL}(n,\mathbb{R}) \cap \mathrm{O}(n,\mathbb{R}) \\
\mathrm{SO}(p,q) &:= \mathrm{SL}(p+q,\mathbb{R}) \cap \mathrm{O}(p,q) \\
\mathrm{SO}(n,\mathbb{C}) &:= \mathrm{SL}(n,\mathbb{C}) \cap \mathrm{O}(n,\mathbb{C}) \\
\mathrm{SU}(n) &:= \mathrm{SL}(n,\mathbb{C}) \cap \mathrm{U}(n)\,.
\end{aligned}$$

Notice that the subgroup N in Example 2.11 is in $\mathrm{SL}(n,\mathbb{R})$. Moreover $A := A_{\det} \cap \mathrm{SL}(n,\mathbb{R})$ is also an important non-trivial subgroup of $\mathrm{SL}(n,\mathbb{R})$.

Example 2.15 Let $\mathcal{H}$ be a complex separable Hilbert space (see Definition A.8). The space of unitary operators of $\mathcal{H}$

$$\mathcal{U}(\mathcal{H}) := \{U\colon \mathcal{H} \to \mathcal{H} : U^{-1} = U^*\}$$

is a topological group with the strong operator topology.

2.2 Compactness and Local Compactness

The Examples 2.1, 2.2 and 2.3 are obviously locally compact. Likewise the Examples 2.4, 2.11 and 2.12 are also locally compact because of Lemma A.2, as well as Example 2.5 if $\mathbb{F}$ is locally compact.

Example 2.16 (See Example 2.6) The homeomorphism group of a topological space X is not necessarily locally compact, even if X is compact **(see Exercise 3.)**.

Example 2.17 (See Example 2.8) Contrary to the homeomorphism group, the isometry group of a metric space X is as "good" as the space itself. In other words, if X is compact, then $\mathrm{Iso}(X)$ is compact and if X is locally compact, then $\operatorname{Iso}(X)$ is locally compact **(Exercise 4.)**. So $\operatorname{Iso}(X)$ is always much much smaller than $\operatorname{Homeo}(X)$.

The proof of the first assertion follows immediately from Ascoli–Arzelà's Theorem (see Theorem A.1). In fact, from the chain of inclusions

$$\operatorname{Iso}(X) \subset \operatorname{Homeo}(X) \subset C(X, X)$$

it follows that $\operatorname{Iso}(X)$ is compact if it is an equicontinuous family, which is obvious since it consists of isometries.

Example 2.18 (See Example 2.11.3. and 2.12) We mentioned already that $\mathrm{O}(p, q)$ is locally compact since it is a closed subgroup of $\mathrm{GL}(p+q, \mathbb{R})$. The question now is whether it is compact and we will show that $\mathrm{O}(p, q)$ is compact if and only if $p = 0$ or $q = 0$.

1. Let $p = 0$ and let $\mathrm{O}(0, n) = \mathrm{O}(n, \mathbb{R})$. Let us write $A \in \mathrm{O}(n, \mathbb{R})$ as $A = ((c_1), \dots, (c_n))$, where $c_j = Ae_j$ for $1 \leq j \leq n$. Thus $\{c_1, \dots, c_n\}$ is an orthonormal basis in $\mathbb{R}^n$. In particular $\|c_j\|^2 = 1$, so that $|A_{ij}| \leq 1$. Thus $\mathrm{O}(n, \mathbb{R})$ is bounded in $\mathbb{R}^{n \times n}$. On the other hand, by definition
$$\mathrm{O}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : \langle Av, Aw \rangle = \langle v, w \rangle = \text{ for all } v, w \in \mathbb{R}^n\} =^{-1} (\mathrm{Id}) ,$$
where $f M_{n \times n} \to M_{n \times m}$ is defined as $f(A) := A^t A/$ so that $\mathrm{O}(n, \mathbb{R})$ is closed and hence compact by the Heine–Borel Theorem.
2. Let us assume now that $p \neq 0$ and we will show that in this case $\mathrm{O}(p, q)$ is not compact since it is not bounded. In fact we can write for $n = p + q$
$$\mathrm{O}(p, q) := \{A \in \mathbb{R}^{n \times n} : Q_p(Av) = Q_p(v) \text{ for all } v \in \mathbb{R}\} ,$$
where
$$Q_p(v) = -\sum_{j=1}^{p} v_j^2 + \sum_{j=p+1}^{n} v_j^2 .$$
Consider for example the case $p = 1$, so that
$$Q_1(v) = -v_1^2 + \sum_{j=2}^{n} v_j^2$$
with respect to the basis $(e_1, \dots, e_n)$. Consider now the change of basis
$$\begin{aligned} e_1' :=& e_2 - e_1 \\ e_2' :=& e_2 + e_1 \\ e_j' :=& e_j \text{ for } 3 \leq j \leq n , \end{aligned}$$

and denote by V the vector space $\mathbb{R}^n$ with this new basis. On V the quadratic form will now take the form

$$Q_1'(v') = -(v_1' - v_2')^2 + (v_1' + v_2')^2 + \sum_{j=3}^{n} {v_j'}^2 .$$

The matrix

$$A_s := \begin{pmatrix} s & 0 & {}^t 0_{n-2} \\ 0 & \frac{1}{s} & {}^t 0_{n-2} \\ 0_{n-2} & 0_{n-2} & \mathrm{Id}_{n-2} \end{pmatrix} \tag{2.4}$$

clearly satisfies

$$Q_1'(A_s v) = Q_1'(v)$$

so that $A_s \in \mathrm{O}(V, Q_1')$, which shows that $\mathrm{O}(V, Q_1')$ is not compact. The general argument for $n > p \geq 1$ is analogous.

Example 2.19 The special linear group $\mathrm{SL}(n, \mathbb{R})$ is a locally compact group since it is closed in $\mathrm{GL}(n, \mathbb{R})$, but it is not compact since the matrix A_t in (2.4) belongs to $\mathrm{SL}(n, \mathbb{R})$ as well.

Example 2.20 (See Example 2.10) Profinite groups are compact.

Example 2.21

1. The *one-dimensional torus*

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

with the usual multiplication is a compact Abelian topological group isomorphic to

$$\mathrm{SO}(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

via the isomorphism

$$\begin{aligned} \mathrm{SO}(2, \mathbb{R}) &\longrightarrow \mathbb{T} \\ X &\longmapsto Xe_1 . \end{aligned}$$

2. The *n-dimensional torus* $\mathbb{T}^n$ is also a compact Abelian topological group.

Example 2.22 We emphasise that $\mathrm{U}(n) \neq \mathrm{O}(n, \mathbb{C})$. In fact:

- $\mathrm{U}(n)$ preserves the usual Hermitian inner product on $\mathbb{C}^n$, Thus

$$\mathrm{U}(n) = \{X \in \mathrm{GL}(n, \mathbb{C}) : {}^*XX = \mathrm{Id}_n\} ,$$

where ${}^*X = {}^t\overline{X}$ and it is compact.

- $\mathrm{O}(n,\mathbb{C})$ preserves a non-degenerate symmetric bilinear form on $\mathbb{C}^n$ so that

$$\mathrm{O}(n,\mathbb{C}) := \{X \in \mathrm{GL}(n,\mathbb{C}) : {}^tXX = \mathrm{Id}_n\}$$

and $\mathrm{O}(n,\mathbb{C})$ is not compact. The argument to see this is exactly the same as for $\mathrm{O}(p,q)$.

Example 2.23 Let $B\colon \mathbb{C}^{2n}\times\mathbb{C}^{2n}\to\mathbb{C}$ be the skew-symmetric bilinear form on $\mathbb{C}^{2n}$ given by $B(x,y) = \sum\limits_{1\le p\le n} x_p y_{n+p} - x_{n+p}y_p$, where $x=(x_1,\dots,x_{2n})$ and $y=(y_1,\dots,y_{2n})$. The symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ is defined as the subgroup of $\mathrm{GL}(2n,\mathbb{C})$ of matrices that leave B invariant. If $F = \begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix}$, then

$$\begin{aligned}\mathrm{Sp}(2n,\mathbb{C}) &:= \{A\in\mathrm{GL}(2n,\mathbb{C}) : B(x,y) = B(Ax,Ay) \text{ for all } x,y\in\mathbb{C}^{2n}\}\\ &= \{A\in\mathrm{GL}(2n,\mathbb{C}) : {}^tAFA = F\}\,.\end{aligned}$$

Related to $\mathrm{Sp}(2n,\mathbb{C})$ there are also the following groups

$$\mathrm{Sp}(2n) := \mathrm{Sp}(2n,\mathbb{C})\cap\mathrm{U}(2n)\,,$$

which is compact, and

$$\mathrm{Sp}(2n,\mathbb{R}) := \mathrm{Sp}(2n,\mathbb{C})\cap\mathrm{GL}(2n,\mathbb{R}) = \{A\in\mathrm{GL}(2n,\mathbb{R}) : {}^tAFA = F\}.$$

Example 2.24 We get back to the space of unitary operators of a complex separable Hilbert space $\mathcal{H}$.

Lemma 2.2

If $\mathcal{H}$ is a complex separable Hilbert space, then the space of continuous unitary operators $\mathcal{U}(\mathcal{H})$ is a topological group that is locally compact if and only if $\dim\mathcal{H}<\infty$, in which case it is compact.

Proof $(\Leftarrow)$ Let us assume that $\dim\mathcal{H} = n<\infty$. Then

$$\mathcal{U}(\mathcal{H}) = \mathrm{U}(n)\,,$$

which is compact.

$(\Rightarrow)$ We prove the assertion by contradiction. A basis neighborhood of $\mathrm{Id}\in\mathcal{U}(\mathcal{H})$ in the strong operator topology is of the form

$$U_{F,\epsilon} := \{T\in\mathcal{U}(\mathcal{H}) : \|Tu-u\|<\epsilon \text{ for all } u\in F\}\,,$$

where $F\subset\mathcal{H}$ is a finite set and $\epsilon>0$. If $\mathcal{U}(\mathcal{H})$ is locally compact, the neighborhood $U_{F,\epsilon}$ is contained in a compact set C. We will show that the assumption that $\mathcal{H}$ is infinite dimensional leads to a contradiction.

We write $\mathcal{H} = \langle F \rangle \oplus \langle F \rangle^{\perp}$. Then an obvious verification shows that the subgroup

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{U}(\langle F \rangle^{\perp}) \end{pmatrix} \subset U_{F,\epsilon}\,,$$

since if $T \in \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{U}(\langle F \rangle^{\perp}) \end{pmatrix}$, then $Tu = u$ for all $u \in F$. But then

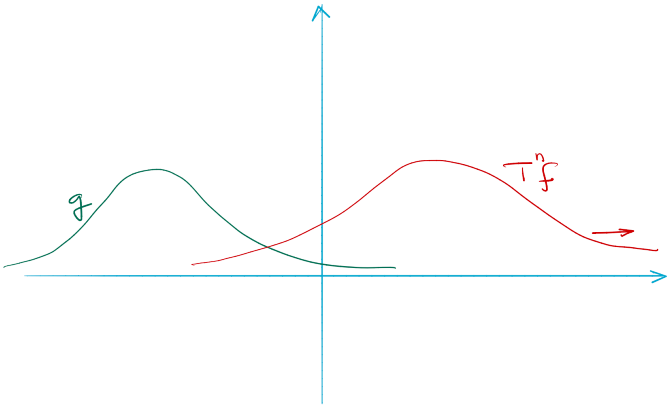
$$\mathcal{U}(\langle F \rangle^{\perp}) \simeq \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{U}(\langle F \rangle^{\perp}) \end{pmatrix} \subset U_{F,\epsilon}\,,$$

that is also $\mathcal{U}(\langle F \rangle^{\perp})$ must be contained in a compact set and hence be compact. But if $F \subset \mathcal{H}$ is finite and $\mathcal{H}$ is infinite dimensional, then $\langle F \rangle^{\perp}$ must be infinite dimensional. We show now that the unitary group of an infinite dimensional Hilbert space cannot be compact.

Claim 2.2.1. *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{U}(\mathcal{H})$ cannot be compact.*

By contradiction let us assume that $\mathcal{U}(\mathcal{H})$ is compact in the strong operator topology. If we can find a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{U}(\mathcal{H})$ of unitary operators converging to zero in the weak operator topology, that is such that $\langle T_n u, v \rangle \to 0$ for all $u, v \in \mathcal{H}$, then by our assumption we could find subsequence $(T_{n_k})_{k\in\mathbb{N}}$ that converges in the strong operator topology to a unitary operator T. On the other hand $(T_{n_k})_{k\in\mathbb{N}}$ converges weakly to zero, which implies that $T = 0$, a contradiction.

Thus, in order to complete the proof, we need to show that if $\mathcal{H}$ is infinite dimensional and separable we can find a sequence of unitary operators $(T_n)_{n\in\mathbb{N}}$ that converges to zero in the weak operator topology. Let $\mathcal{H} = L^2(\mathbb{R})$ and let $T \in L^2(\mathbb{R})$ be the translation by one, $Tf(x) := f(x-1)$. Then if $f, g \in L^2(\mathbb{R})$, $\langle T^n f, g \rangle \to 0$. In fact, the space of C^∞ compactly supported functions on $\mathbb{R}$ is dense in $L^2(\mathbb{R})$ and $\langle T^n f, g \rangle$ is very small as soon as n is large enough that the supports of $T^n f$ and of g are almost disjoint.



□

2.3 General Facts about Topological Groups

The simple fact of requiring that the group operations are continuous has a plethora of interesting consequences, of which we illustrate here the most important ones.

Definition 2.2. Symmetric neighborhood

A neighborhood U of the identity $e \in G$ in a topological group is symmetric *if $g^{-1} \in U$ whenever $g \in U$.* ♣

Proposition 2.1

Let G be a topological group. Then

1. *If V is a neighborhood of the identity $e \in G$, there exists a symmetric neighborhood U of the identity contained in V.*
2. *If V is a neighborhood of the identity $e \in G$, there exists a symmetric neighborhood U of the identity such that $U^2 = U^{-1}U \subset V$.*
3. *If $H < G$ is a subgroup, then its closure $\overline{H}$ is also a subgroup*
4. *If G is connected any discrete normal subgroup is central.*
5. *The connected component G° of the identity is a closed normal subgroup.*
6. *Every open subgroup is closed.*
7. *If G is connected and U is any neighborhood of the identity $e \in G$, then $G = \cup_{n=1}^{\infty} U^n$.* ♠

Note that the converse of Proposition 2.1.6. is not true. For example $\mathbb{R} < \mathbb{R}^2$ is a closed subgroup that is not open.

Proof (1) is immediate by taking $U := V \cap V^{-1}$ and (2) is also immediate from the continuity of the multiplication and from (1).

(3) Since the multiplication and the inversion are continuous, then

$$\begin{aligned} m(\overline{H} \times \overline{H}) &= m(\overline{H \times H}) \subseteq \overline{m(H \times H)} = \overline{H} \\ i(\overline{H}) &\subseteq \overline{H}\,. \end{aligned}$$

(4) Let D be a discrete normal subgroup and, for $h \in D$ fixed, let us define the continuous map $\gamma_h \colon G \to D$ by $\gamma_h(g) := ghg^{-1}$. We want to show that $\gamma_h(g) \equiv h$ and this will follow from the connectedness of G and the discreteness of D. In fact, since G is connected, γ_h is continuous and D is discrete, then $\mathrm{image}(\gamma_h)$ must be one point. Since $\gamma_h(e) = ehe^{-1} = h$, then $\gamma_h(g) = h$ for all $g \in G$. Thus $ghg^{-1} = h$ for all $g \in G$, so that $gh = hg$ for all $g \in G$, that is D is central.

(5) Let G° be the connected component of the identity $e \in G$. Since the multiplication $m \colon G^\circ \times G^\circ \to G$ is continuous and $G^\circ \times G^\circ$ is connected, then $m(G^\circ \times G^\circ)$ is connected. But $e \in m(G^\circ \times G^\circ)$, so that $m(G^\circ \times G^\circ) \subset G^\circ$, that is G° is closed under multiplication. Likewise the image of $i \colon G^\circ \to G^\circ$ is connected and contains e, so that $i(G^\circ) \subset G^\circ$. Thus G° is a group.

To see that G° is closed, observe that $G^\circ \subset \overline{G^\circ}$. But $\overline{G^\circ}$ is connected and contains the identity in G so that $\overline{G^\circ} \subset G^\circ$. Thus $\overline{G^\circ} = G^\circ$.

If $g \in G$, consider now the continuous map defined by the conjugation $c_g \colon G^\circ \to G$, $c_g(h) = ghg^{-1}$. Since G° is connected, $c_g(G^\circ)$ is connected, hence contained in G°, which means that G° is normal.

(6) Let $H < G$ be an open subgroup. If $L_g \colon G \to G$ is the left multiplication by $g \in G$, by continuity of the multiplication $L_g H$ is also open for all $g \in G$. Thus the union $G \setminus H = \cup gH$ over all $g \in G \setminus H$ is open and hence H is closed.

(7) Obviously $\cup_{n=1}^\infty U^n \subseteq G$. Let $V \subset U$ be an open symmetric neighborhood of $e \in G$ such that $V^2 = V^{-1}V \subset U$. Then $H := \cup_{n=1}^\infty V^n \subseteq \cup_{n=1}^\infty U^n \subseteq G$ is an open subgroup of G, hence closed by (6). Since G is connected, we have equality. $\square$

2.4 Local homomorphisms

The content of this section will be heavily used in the correspondence between Lie groups and Lie algebras presented in § **??** and it is of independent interest.

Definition 2.3. Local homomorphism

Let G, H be topological groups.

1. *A* local homomorphism *is a continuous map $\varphi : U \to H$, where U is a neighborhood of $e \in G$, such that whenever $x, y, xy \in U$*
$$\varphi(xy) = \varphi(x)\varphi(y).$$
2. *A local homomorphism $\varphi\colon U \to H$ is a* local isomorphism *if it is bijective onto $\varphi(U)$ and $\varphi^{-1}\colon \varphi(U) \to G$ is continuous.*

♣

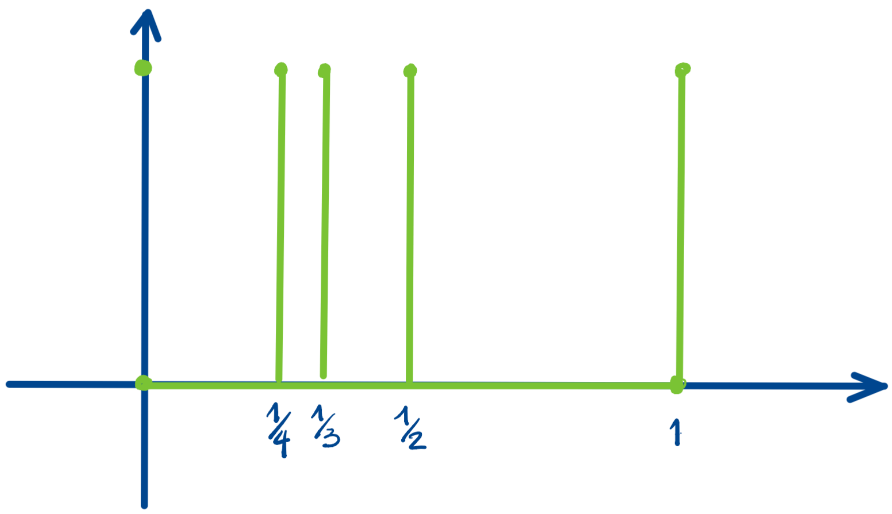
A natural question to ask is when a local homomorphism φ of a topological group can be extended to a homomorphism.

Theorem 2.1. Extension of local homomorphisms

If G is a simply connected topological group, then any local homomorphism extends uniquely to a homomorphism $G \to H$.

♡

Recall that a topological space X is *simply connected* if it is path-connected and $\pi_1(X)$ is trivial. Path-connectedness implies connectedness but the converse is not true in general. For example, let $X := [0, 1] \times \{0\} \cup \{\{\frac{1}{n}\} \times [0, 1] : n \in \mathbb{N}\} \cup \{\{0\} \times [0, 1] \setminus \{0\} \times (0, 1)\}$.



Then X is connected but not path-connected.

However, if a space is connected and locally path-connected, then it is path-connected. For example connected manifolds, and in particular connected Lie groups, are automatically path-connected, since they are locally homeomorphic to $\mathbb{R}^n$, which is path-connected.

Proof We give only the sketch of the proof. For the complete argument see [4]. Let $U \subset G$ be a neighborhood of $e \in G$ and $\varphi\colon U \to H$ the local homomorphism that we want to extend. We will prove the theorem in three steps:

1. We use that G is path-connected to define φ on all of G.
2. We use that $\pi_1(G) = 0$ to show that the extension is well-defined.
3. We show that φ is the unique continuous extension of $\varphi|_U$..

1. Since G is path-connected, if $g \in G$, let $\alpha : [0,1] \to G$ be a path from e to g. Choose a partition of $[0,1]$ into subintervals $I_k := [t_{k-1}, t_k]$, for $k = 1, \dots, n$, with the property that if $s, t \in I_k$, then

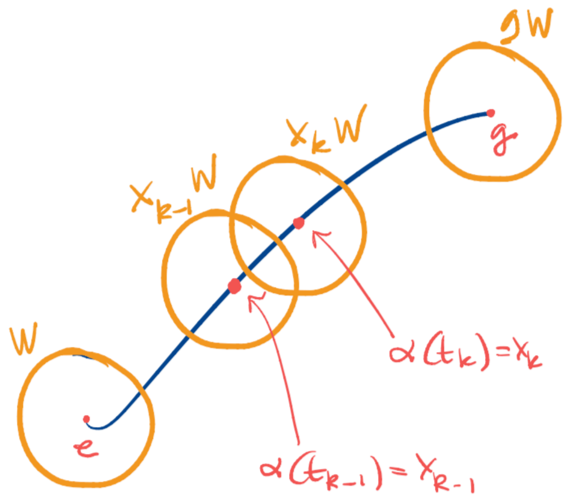
$$\alpha(s)^{-1}\alpha(t) \in U\,.$$

We call such a partition *good*. We impose a further condition that will be needed only in Step 2., but that we may as well impose from the beginning. We choose W be a neighborhood of $e \in G$ contained in U such that $W = W^{-1}$ and $W^2 \subset U$ and $\alpha \subset \cup_{k=0}^n \alpha(t_k)W$. Such a partition exists since $[0,1]$ is compact and the group operations are continuous, so that there exists a $\delta > 0$ such that $\alpha(s)^{-1}\alpha(t) \in U$ whenever $|s-t| < \delta$. Set $x_k := \alpha(t_k) \in \alpha$, with $x_0 = \alpha(0) = e$ and

$g = \alpha(t_n) = x_n$. Then

$$g = (x_0^{-1}x_1)(x_1^{-1}x_2)\dots(x_{n-1}^{-1}x_n)\,,$$

with $x_{k-1}^{-1}x_k \in U$.



Then we define

$$\varphi_\alpha(g) := \varphi(x_0^{-1}x_1)\varphi(x_1^{-1}x_2)\dots\varphi(x_{n-1}^{-1}x_n)\,,$$

To show that $\varphi_\alpha(g)$ is independent of the partition, notice that adding points to the partition gives a partition that still has the above defining properties. Let us hence take $t \in I_k$ and write $[t_{k-1}, t_k] = [t_{k-1}, t] \cup [t, t_k]$. Since $t \in I_k$, then $\alpha(t_{k-1})^{-1}\alpha(t) \in U$, $\alpha(t)^{-1}\alpha(t_k) \in U$ and

$$\alpha(t_{k-1})^{-1}\alpha(t_k) = \alpha(t_{k-1})^{-1}\alpha(t)\alpha(t)^{-1}\alpha(t_k) \in U\,,$$

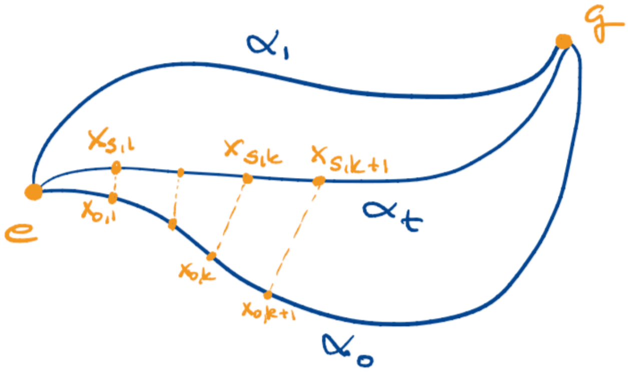
so that

$$\varphi_\alpha(\alpha(t_{k-1})^{-1}\alpha(t_k)) = \varphi(\alpha(t_{k-1})^{-1}\alpha(t))\varphi(\alpha(t)^{-1}\alpha(t_k))\,.$$

2. We show now that $\varphi_\alpha(g)$ is in fact independent of α. Since $\pi_1(G) = 0$ we can choose a homotopy $H\colon [0,1] \times [0,1] \to G$ with $H(0,t) = \alpha_0(t)$ and $H(1,t) = \alpha_1(t)$ and set $\varphi_s := \varphi_{\alpha_s}$, where $\alpha_s\colon [0,1] \to G$ is defined as $\alpha_s(t) := H(s,t)$. Let $\delta > 0$ be such that

$$H(s_1,t_1)^{-1}H(s_2,t_2) \in W$$

for all $s_1, s_2, t_1, t_2 \in [0,1]$ with $|s_1 - s_2| + |t_1 - t_2| < \delta$. Then for all $s \in [0,1]$, the partition $\{t_k\}_{k=0}^n := \{\frac{k}{n}\}_{k=0}^n$ is good, where we choose n large enough that $\frac{1}{n} < \frac{\delta}{2}$.



Let

$$A := \{s \in [0,1] : \varphi_s(g) = \varphi_0(g)\} .$$

Since $0 \in A \neq \varnothing$, it will be enough to show that A is open and closed.

To see that A is closed, we will show that if $(s_j)_{j\in\mathbb{N}} \subset A$ and $s_j \to s$ for $j \to \infty$, then $s \in A$. Let α_{s_j} and α_s be the corresponding paths and let $\{t_k\}_{k=0}^n$ be the good partition of $[0,1]$ chosen above. By continuity of H, one deduces that

$$\lim_{j\to\infty} \alpha_{s_j}(t_k) = \alpha_s(t_k) .$$

Writing $x_{s_j,k} := \alpha_{s_j}(t_k)$ and $x_{s,k} := \alpha_s(t_k)$, and using that the φ_{s_j} are continuous, one deduces that

$$\lim_{j\to\infty} \varphi(x_{s_j,k-1}^{-1} x_{s_j,k}) = \varphi_s(x_{s,k-1}^{-1} x_{s,k}) .$$

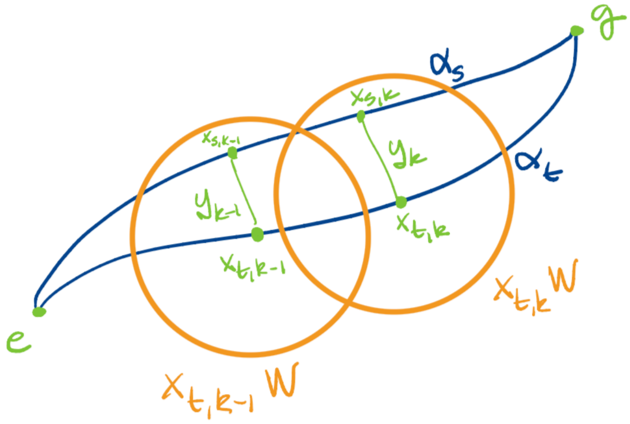
Thus

$$\lim_{j\to\infty} \varphi_{s_j}(g) = \lim_{j\to\infty} \prod_{k=0}^{n-1} \varphi(x_{s_j,k}^{-1} x_{s_j,k+1}) = \prod_{k=0}^{n-1} \varphi_s(x_{s,k}^{-1} x_{s,k+1}) .$$

Since each term on the left hand side is equal to $\varphi_0(g)$, so is the one on the right hand side.

To see that A is open, let $t \in A$ and let $s \in [0,1]$ be close enough to t so that $\alpha_s \subset \cup_{j=0}^n x_j W$, where $x_j := \alpha(t_j)$ were defined at the beginning of the proof. We can define $y_k := x_{s,k}^{-1} x_{t,k} \in W$ so that

$$x_{s,k-1}^{-1} x_{s,k} = y_{k-1} x_{t,k-1}^{-1} x_{t,k} y_k^{-1} ,$$



and

$$\varphi_s(g) = \prod_{k=1}^{n} \varphi(x_{s,k-1}^{-1} x_{s,k}) = \prod_{k=1}^{n} \varphi(y_{k-1} x_{t,k-1}^{-1} x_{t,k} y_k^{-1}) = \prod_{k=1}^{n} \varphi(x_{t,k-1}^{-1} x_{t,k}) = \varphi_t(g)\,.$$

Thus $s \in A$, that is A is open.

3. It is easy to see that φ is continuous. To see that it is a homomorphism, let α be a path from e to g and β a path from e to h. Then the concatenation of α with $g\beta$ is a path from e to gh and by definition $\varphi(gh) = \varphi(g)\varphi(h)$. The uniqueness follows immediately from Proposition 2.1.7. □

2.5 Haar Measure and Homogeneous Spaces

2.5.1 Haar Measure

Let X be a locally compact topological space and G a topological group. A *left action* of G on X by homeomorphisms is a homomorphism $G \to \mathrm{Homeo}(X)$, that is a map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

such that $(g_2 g_1, x) = (g_2, (g_1, x))$ for all $g_1, g_2 \in G$ and $x \in X$. The action is continuous if $G \times X \to X$ is a continuous map, in which case

$$\begin{aligned} \varphi_g \colon X &\longrightarrow X \\ x &\longmapsto gx \end{aligned}$$

is a homeomorphism with inverse $\varphi_{g^{-1}}$. If $C_c(X)$ is the space of continuous functions with compact support and G acts on X, then there is a continuous representation $\lambda\colon G \to \mathrm{Iso}(C_c(X))$

defined by $(\lambda(g)f)(x) := f(g^{-1}x)$ (see Lemma A.3). Likewise G acts continuously on the left on the space $C_c(X)^*$ of continuous linear functionals on $C_c(X)$, via the contragredient representation $\lambda^*(g)(\Lambda))(f) := \Lambda(\lambda(g^{-1})f)$.

Remark A right G-action on X $(g,x) \longmapsto xg$ would induce a right G-action on $C_c(X)$, $(\rho(g)f)(x) := f(xg)$ and hence on $C_c(X)^*$, $(\rho^*(g)(\Lambda))(f) := \Lambda(\rho(g)(f))$.

A left (resp. right) action is an action for which, given the product g_1g_2 acting on X, first g_2 acts (resp. g_1) followed then by g_1 (resp. g_2).

Theorem 2.2. Riesz Representation Theorem

Let X be a locally compact Hausdorff topological space. If Λ is a positive linear functional on $C_c(X)$ (that is $\Lambda(f) \geq 0$ if $f \in C_c(X)$ with $f \geq 0$), then there exists a unique regular Borel measure μ on X that represents Λ, that is such that for every $f \in C_c(X)$,

$$\Lambda(f) = \int_X f(x)\,d\mu(x)\,.$$

♡

(For the definition of *regular Borel measure* see Definition A.7.)

Notice that the action on the left of a group G on Λ is reflected in the action on the measure given by the contragredient action and the identification of functionals with regular Borel measures given by Riesz Representation Theorem. In other words the G-action on measures on $C_c(X)$ is denoted by $(g,\mu) \mapsto g_*\mu$, where

$$(g_*\mu)(A) := \mu(g^{-1}A)\,,$$

so that

$$(\lambda(g)^*\Lambda)(f) = \int_X f(gx)\,d\mu(x) = \int_X f(x)\,d(g_*\mu)(x) = \int_X f(x)\,d\mu(g^{-1}x)\,.$$

A particular action is the one of a locally compact Hausdorff group on itself.

Definition 2.4. (Haar measure)

A left *(resp.* right*)* Haar measure *on a locally compact Hausdorff group G is a non-zero positive linear functional*

$$m\colon C_c(G) \to \mathbb{C}$$

that is invariant under left (resp. right) translation, that is such that

$$(g_*m)(f) = m(f)$$

for all $f \in C_c(G)$. ♣

In the following we will use the notations $m(f)$, $\int_G f(x)$, $dm(x)$ or dx according to what we want to emphasize or for simplicity.

Theorem 2.3. (Existence and Uniqueness of the Haar measure, 1933)

A left (resp. right) Haar measure on a locally compact Hausdorff group always exists and is unique up to positive multiplicative constants. ♡

We will verify the uniqueness. However the proof of the existence of the Haar measure in general is long, technical and does not bring much insight. There are however cases in which the proof is simple and follows on standard yet useful techniques. This is the case for example for compact groups (see [12, Theorem 2.2.3]) or for Lie groups **(see ??)**.

Lemma 2.3

Let m *be a left Haar measure. If* $f \in C_c(G)$ *and* $x \in G$*, let* $\check{f}(x) := f(x^{-1})$*. Then* $n(f) := m(\check{f})$ *is a right Haar measure.* ♡

Proof We need to verify that $n(\rho(g)(f)) = n(f)$ for every $g \in G$ and for every $f \in C_c(G)$. Notice that

$$(\rho(g)\check{f})(x) = (\rho(g)f)(x^{-1}) = f(x^{-1}g)$$

so that

$$\begin{aligned} n(\rho(g)f) = m((\rho(g)f)\check{\ }) &= \int_G f(x^{-1}g)\,dm(x) \\ &= \int_G \check{f}(g^{-1}x)\,dm(x) = \int_G \check{f}(x)\,dm(x) = n(f)\,. \end{aligned}$$

□

Lemma 2.4

Let G *be a locally compact Hausdorff group with left Haar measure* m*. Then*

1. $\operatorname{supp}(m) = G$*, and*
2. *If* $h \in C(G)$ *is such that*
$$\int_G h(x)\varphi(x)\,dm(x) = 0$$
for all $\varphi \in C_c(G)$*, then* $h \equiv 0$. ♡

Proof (1) Recall that $\operatorname{supp}(m) := \{x \in G :$ for every open set U containing x, $m(U) > 0\}$.

Since $m \not\equiv 0$, there exists $f \in C_c(G)$ such that $m(f) > 0$. Let $K := \operatorname{supp}(f)$ with $m(K) > 0$. If $G \neq \operatorname{supp}(m)$, then there exists $x \in G \smallsetminus \operatorname{supp}(m)$ and an open neighborhood $U \ni x$ with $m(U) = 0$. But a finite number of translates of U would cover K, so that $m(K) = 0$, which is a contradiction.

(2) We show that $h(e) = 0$ (which is anyway all we need in the proof of the uniqueness of the Haar measure) and the argument for any other point follows by translation. Let $\epsilon > 0$. By continuity of h there exists an open neighborhood $V \ni e$ such that for all $g \in V$

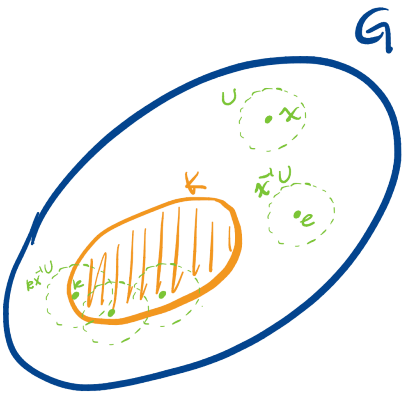
$$|h(g) - h(e)| < \epsilon\,.$$

By Urysohn's lemma there exists $\varphi \in C_c(G)$ such that $\varphi \geq 0$, $\varphi(e) > 0$ and $\operatorname{supp}(\varphi) \subset V$. Since $\int_G h(g)\varphi(g)\,dm(g) = 0$ for all $\varphi \in C_c(G)$, then

$$\begin{aligned}
&|h(e)|\left|\int_G \varphi(g)\,dm(g)\right| \\
&= \left|\int_G h(e)\varphi(g)\,dm(g)\right| \\
&= \left|\int_G h(g)\varphi(g)\,dm(g) - \int_G h(e)\varphi(g)\,dm(g)\right| \\
&\leq \int_G |h(g) - h(e)|\varphi(g)\,dm(g) \\
&\leq \epsilon \int_G \varphi(g)\,dm(g)\,,
\end{aligned}$$

from which it follows that $|h(e)| < \epsilon$ for all $\epsilon > 0$, that is $h(e) = 0$. □

We remark that we used in the first part of the proof that G is a topological group. In fact, the fact that we can cover K with translates of a neighborhood U of $x \in G \smallsetminus K$ is only possible because we are in a topological group.



Proof [Proof of the uniqueness of the Haar measure in Theorem 2.3] Let m be an arbitrary left Haar measures and n an arbitrary right Haar measure (which exists by Lemma 2.3). Let $f, g \in C_c(G)$

be such that $m(f) \neq 0$ (this certainly exists since m is non-zero).

$$\begin{aligned} m(f)n(g) &= m(f)\int_G g(y)\,dn(y) \\ &\overset{(1)}{=} m(f)\int_G g(yt)\,dn(y) \\ &= \int_G f(t)\left(\int_G g(yt)\,dn(y)\right)dm(t) \\ &\overset{(2)}{=} \int_G\left(\int_G f(t)g(yt)\,dm(t)\right)dn(y) \\ &\overset{(3)}{=} \int_G\left(\int_G f(y^{-1}x)g(x)\,dm(x)\right)dn(y) \\ &\overset{(4)}{=} \int_G\left(\int_G f(y^{-1}x)\,dn(y)\right)g(x)\,dm(x)\,, \end{aligned}$$

where we used:

– in (1) that n is right invariant;

– in (2) and in (4) Fubini;

– in (3) the right invariance of n, the left invariance of m and we set $x = yt$.

Note that we could use Fubini's theorem since the support of all functions is compact and hence

$$\int_{G\times G} |f(t)g(yt)|\,dm(t)\,dn(y) < \infty$$

and

$$\int_{G\times G} |f(y^{-1}x)g(x)|\,dm(x)\,dn(y) < \infty$$

Let us now define $w_f\colon G \to \mathbb{R}$ by

$$w_f(x) := \frac{1}{m(f)}\int_G f(y^{-1}x)\,dn(y)\,,$$

so that

$$n(g) = \frac{1}{m(f)}\int_G\left(\int_G f(y^{-1}x)\,dn(y)\right)g(x)\,dm(x) = \int_G w_f(x)g(x)\,dm(x)\,.$$

Since the left hand side is independent of f, for all $f_1, f_2 \in Cc(X)$ with $m(f_i) \neq 0$, $i = 1, 2$, then

$$\int_G w_{f_1}(x)g(x)\,dm(x) - \int_G w_{f_2}(x)g(x)\,dm(x) = 0\,.$$

Since $w_{f_1} - w_{f_2}$ is continuous, by Lemma 2.4 $w_f(e)$ is independent of f, so that $w_f(e) = C$ for some $C \in \mathbb{R}$. Thus

$$m(f)C = m(f)w_f(e) = m(f)\frac{1}{m(f)}\int_G f(y^{-1})\,dn(y) = \int_G f(y^{-1})\,dn(y) = n(\check{f})\,.$$

If now we choose $n(f) := m'(\check{f})$, which is a well-defined left Haar measure by Lemma 2.3, then

$$m(f)C = m'(f)$$

for all $f \in C_c(G)$ such that $m(f) \neq 0$. □

Example 2.25

1. The Lebesgue measure on $(\mathbb{R}^n, +)$ is the left and right Haar measure.
2. The Lebesgue measure on $G := (\mathbb{R}_{>0}, \cdot)$ is neither left nor right invariant, but
$$f \mapsto \int_G f(x)\,\frac{dx}{x}$$
defines both the left and the right Haar measure on G.
3. If G is discrete, then the counting measure is both a left and a right Haar measure.

The above examples bring to the question as to when a left Haar measure is also right invariant. To approach this question let $\mathrm{Aut}(G)$ be the group of continuous invertible automorphisms of G with continuous inverse. Then $\mathrm{Aut}(G)$ acts on $C_c(G)$ on the left via

$$(\alpha \cdot f)(x) := f(\alpha^{-1}(x))$$

for $\alpha \in \mathrm{Aut}(G)$, $f \in C_c(G)$ and $x \in G$. If m is a left Haar measure on G, one can easily verify that the linear form

$$f \mapsto m(\alpha \cdot f)$$

is also a left Haar measure. In fact

$$\begin{aligned} m(\alpha \cdot \lambda(g)(f)) &= \int_G (\alpha \cdot \lambda(g)(f))(x)\, dm(x) = \int_G (\lambda(g)f)(\alpha^{-1}(x))\, dm(x) \\ &= \int_G f(\alpha^{-1}(g^{-1}x))\, dm(x) = \int_G (\alpha \cdot f)(x)\, dm(x) = m(\alpha \cdot f) \end{aligned}$$

Thus there exists a positive constant $\mathrm{mod}_G(\alpha)$ such that

$$m(\alpha \cdot f) = \mathrm{mod}_G(\alpha) m(f)\,. \tag{2.5}$$

Lemma 2.5

The function $\mathrm{mod}_G \colon \mathrm{Aut}(G) \to (\mathbb{R}_{>0}, \cdot)$ *is a homomorphism.* ♡

Proof Since $\mathrm{Aut}(G)$ acts on $C_c(G)$ on the left, then $(\alpha\beta) \cdot f = \alpha \cdot (\beta \cdot f)$. Then for all $f \in C_c(G)$

$$\begin{aligned} \mathrm{mod}_G(\alpha\beta) m(f) &= m((\alpha\beta) \cdot f) = m(\alpha \cdot (\beta \cdot f)) \\ &= \mathrm{mod}_G(\alpha) m(\beta \cdot f) \\ &= \mathrm{mod}_G(\alpha)\mathrm{mod}_G(\beta) m(f)\,. \end{aligned}$$

□

Let now consider the conjugation automorphism $\alpha = c_g$, for $g \in G$,

$$\begin{aligned} c_g \colon G &\to G \\ x &\mapsto gxg^{-1}\,, \end{aligned} \tag{2.6}$$

so that $(c_g \cdot f)(x) = f(g^{-1}xg)$. We use the notation $\Delta_G(g) := \mathrm{mod}_G(c_g)$ and we call $\Delta_G \colon G \to (\mathbb{R}_{>0}, \cdot)$ the *modular function of* G. Explicitly the formula (2.5) for $\alpha = c_g$ gives

$$\begin{aligned} \Delta_G(g)m(f) = m(c_g \cdot f) &= \int_G (c_g \cdot f)(x)\, dm(x) = \int_G f(g^{-1}xg)\, dm(x) \\ &= \int_G f(xg)\, dm(x) = m(\rho(g)f) \end{aligned}$$

so that

$$m(\rho(g)f) = \Delta_G(g)m(f)\,, \tag{2.7}$$

which shows that the modular function captures the extent to which a given left Haar measure fails to be right invariant.

Proposition 2.2

Let G be a locally compact Hausdorff topological group with left Haar measure m and let $\Delta_G \colon G \to \mathbb{R}_{>0}$ be its modular function. Then

1. *Δ_G is continuous and*
2. *for every $f \in C_c(G)$*
$$\int_G f(x^{-1})\Delta_G(x)\, dm(x) = \int_G f(x)\, dm(x)\,.$$

♠

Proof (1) Since $\rho \colon G \to \mathrm{Iso}(C_c(G))$ is continuous when $\mathrm{Iso}(C_c(G))$ is given the strong operator topology, then

$$\lim_{x \to y} \|\rho(x)f - \rho(y)f\|_\infty = 0$$

for all $f \in C_c(G)$ and all $x, y \in G$. It follows that

$$0 = \lim_{x \to y} |m(\rho(x)f) - m(\rho(y)f)| = \lim_{x \to y} |m(f)|\, |\Delta_G(x) - \Delta_G(y)|\,,$$

that is Δ_G is continuous.

(2) Let us set $f^*(x) := f(x^{-1})\Delta_G(x)$ and let us observe that

$$\begin{aligned}(\lambda(g)f)^*(x) &= (\lambda(g)f)(x^{-1})\Delta_G(x) = f(g^{-1}x^{-1})\Delta_G(x)\\ &= \Delta_G(g)^{-1}f(g^{-1}x^{-1})\Delta_G(xg) = \Delta_G(g)^{-1}f^*(xg)\\ &= \Delta_G(g)^{-1}(\rho(g)f^*)(x)\,.\end{aligned}$$

Notice that $m'(f) := m(f^*)$ is also a left Haar measure. In fact,

$$m((\lambda(g)f)^*) = m(\Delta_G(g)^{-1}(\rho(g)f^*)) = \Delta_G(g)^{-1}m(\rho(g)f^*) = \Delta_G(g)^{-1}\Delta_G(g)m(f^*) = m(f^*)$$

Thus there exists $C > 0$ such that $m'(f) = Cm(f)$ and we want to show that $C = 1$. Since Δ_G is continuous, for every $\epsilon > 0$ there exists a symmetric neighborhood $V \ni e$ such that

$$|\Delta_G(x) - 1| < \epsilon$$

for every $x \in V$. Let $f \in C_c(G)$ be a symmetric function with support in V and such that $m(f) = 1$. Then for every $\epsilon > 0$

$$\begin{aligned}|1 - C| &= |(1 - C)m(f)| = |m(f) - m'(f)| = |m(f) - m(f^*)|\\ &\overset{(*)}{=} |m(f) - m(\Delta_G f) = |m((1 - \Delta_G)f)| < \epsilon\, m(f) = \epsilon\,,\end{aligned}$$

where in $(*)$ we used that f is symmetric. □

Definition 2.5

A group G is unimodular *if $\Delta_G \equiv 1$, that is if the left Haar measure and the right Haar measure coincide.* ♣

Since for a left Haar measure m we have by Lemma 2.3 that $m(\check{f})$ is a right Haar integral, the following is immediate

Corollary 2.1

The Haar measure of a group G is inverse invariant if and only if the group is unimodular. ♡

Example 2.26

1. Any locally compact Hausdorf Abelian group is unimodular.
2. Any discrete group is unimodular, since the Haar measure is just the counting measure.
3. Since there are no non-trivial compact subgroups of $(\mathbb{R}_{>0}, \cdot)$, any compact group is unimodular.
4. We show that $\mathrm{GL}(n, \mathbb{R})$ is unimodular. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, we consider the restriction $dm(X) := \prod_{i,j=1}^{n} dX_{i,j}$ to $\mathrm{GL}(n, \mathbb{R})$ of the Lebesgue measure on $\mathbb{R}^{n\times n}$, where $X = (X_{i,j})_{i,j=1}^{n}$. We claim that $|\det X|^{-n}\, dm(X)$ is both a left

and a right Haar measure on $\mathrm{GL}(n,\mathbb{R})$. In fact, let $T_g\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be defined by $T_g(X) := gX$ and let us observe, by writing $X = ((v_1),(v_2),\dots,(v_n))$, that $|\det(dT_g)| \equiv |\det g|^n$. Thus

$$\begin{aligned}
&\int_{\mathrm{GL}(n,\mathbb{R})} (\lambda(g)f)(X)|\det X|^{-n}\,dm(X)\\
&=\int_{\mathrm{GL}(n,\mathbb{R})} f(g^{-1}X)|\det X|^{-n}\,dm(X)\\
&=|\det g|^{-n}\int_{\mathrm{GL}(n,\mathbb{R})} f(g^{-1}X)|\det(g^{-1}X)|^{-n}\,dm(X)\\
&=|\det g|^{-n}\int_{\mathrm{GL}(n,\mathbb{R})} f(X)|\det(X)|^{-n}|\,|\det(dT_{g^{-1}}(X))|^{-n}\,dm(X)\\
&=|\det g|^{-n}\int_{\mathrm{GL}(n,\mathbb{R})} f(X)|\det(X)|^{-n}|\,|\det g^{-1}|^{-n}\,dm(X)\\
&=\int_{\mathrm{GL}(n,\mathbb{R})} f(X)|\det X|^{-n}\,dm(X)\,.
\end{aligned}$$

A similar calculation shows the right invariance.

5. We consider the group $\mathbb{R}_{>0}\ltimes_\eta\mathbb{R}$, where $\eta\colon\mathbb{R}_{>0}\to\mathrm{Aut}(\mathbb{R})$ is defined by $\eta(a)(b) := ab$, so that the product is $(a,b)(a',b') = (aa', b+ab')$. Then $\mathbb{R}_{>0}\ltimes_\eta\mathbb{R}$ is the group of affine transformations of the real line, $(a,b)x = ax+b$, where $a\in\mathbb{R}_{>0}$ and $b\in\mathbb{R}$ and can be identified with the group
$$\left\{\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix} : a\in\mathbb{R}_{>0}, b\in\mathbb{R}\right\}$$
acting on $\mathbb{R}\simeq\{(x,0) : x\in\mathbb{R}\}\subset\mathbb{R}^2$. It is easy to verify that $\frac{da}{a^2}db$ is a left Haar measure and that $\frac{da}{a}db$ is a right Haar measure, so that $\mathbb{R}_{>0}\ltimes_\eta\mathbb{R}$ is not unimodular.
6. We consider the Heisenberg group $\mathbb{R}\ltimes_\eta\mathbb{R}^2$, where $\eta\colon\mathbb{R}\to\mathrm{Aut}(\mathbb{R}^2)$ is defined by
$$\eta(x)\begin{pmatrix} y\\ z\end{pmatrix} := \begin{pmatrix} y\\ z+xy\end{pmatrix}$$
for $x\in\mathbb{R}$, $\begin{pmatrix} y\\ z\end{pmatrix}\in\mathbb{R}^2$, so that the group operation is
$$\left(x_1,\begin{pmatrix} y_1\\ z_1\end{pmatrix}\right)\left(x_2,\begin{pmatrix} y_2\\ z_2\end{pmatrix}\right) = \left(x_1+x_2,\begin{pmatrix} y_1+y_2\\ z_1+z_2+x_1y_2\end{pmatrix}\right).$$

It is easy to see that it can be identified with the group

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

and that the Lebesgue measure is both the left and the right Haar measure, so that $\mathbb{R} \ltimes_\eta \mathbb{R}^2$ is unimodular.

7. The group

$$P := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\} \tag{2.8}$$

is not unimodular since $\frac{da}{a^2} db$ is a left Haar measure and $da\, db$ is a right Haar measure.

8. Any closed normal subgroup of a unimodular group is unimodular. This follows from the following proposition.

Proposition 2.3

Let G be a locally compact Hausdorff group and let $H \trianglelefteq G$ be a closed normal subgroup. Then $\Delta_H = \Delta_G|_H$. Thus if G is unimodular, H is also unimodular.

We will prove this later. For the moment we remark that it is essential that H is normal. In fact, for example $\mathrm{GL}(2, \mathbb{R})$ is unimodular, but the subgroup P in (2.8) is not.

Proposition 2.4

Let G be a locally compact Hausdorff topological group with left Haar measure m. Then $m(G) < \infty$ if and only if G is compact.

Proof ($\Leftarrow$) Since G is compact, the function identically equal to 1 is in $C_c(G)$ Thus $m(G) = m(1) < \infty$.

($\Rightarrow$) Let us assume that G is not compact. We will cover G with an infinite number of disjoint translates of a neighborhood of the identity (hence of positive measure by Lemma 2.4), thus showing that $m(G) = \infty$.

Let $U \subset G$ be a neighborhood of the identity with compact closure. Since G is not compact, it cannot be covered by a finite number of translates of U. Thus there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset G$ such that $x_1 = e$ and $x_n \notin \cup_{j=1}^{n-1} x_j U$. Let $V \subset G$ be a symmetric neighborhood of the identity such that $VV^{-1} \subset U$. Then $x_n \notin \cup_{j=1}^{n-1} x_j VV^{-1}$ and we claim that if $k \neq n$, then $x_n V \cap x_k V = \varnothing$.

In fact, if $x \in x_nV \cap x_kV$, there exists $v_1, v_2 \in V$ such that $x = x_nv_1 = x_kv_2$. But, without loss of generality, if $n > k$ this implies that $x_n \in x_kVV^{-1} \subset \cup_{j=1}^{n-1} x_jVV^{-1}$, which contradicts the definition of (x_n). Thus $G \supset \dot{\cup}_{j=1}^{\infty} x_jV$, so that

$$m(G) \geq m\left(\sum_{j=1}^{\infty} x_jV\right) = \sum_{j=1}^{\infty} m(x_jV) = \sum_{j=1}^{\infty} m(V) = \infty\,,$$

where the last equality comes from the fact that $m(V) > 0$. □

If G is compact, the Haar measure is usually normalized so that $m(G) = 1$.

2.5.2 Homogeneous Spaces of Topological Groups

Let G be a group and $H < G$ a subgroup. Then G acts on the homogeneous space G/H on the left by translations $(g, g'H) \mapsto gg'H$ and the projection $p\colon G \to G/H$ is a G-map, that is it commutes with the G-action on G and on G/H. If G and H are topological groups, we endow G/H with the quotient topology, that is $U \subset G/H$ is open if and only if $p^{-1}(U) \subset G$ is open. This is the finest topology that makes p continuous.

Proposition 2.5

Let $H \leq G$ be topological groups. Then:

1. *The projection p is open, that is it sends open sets into open sets.*
2. *The action of G on G/H is continuous.*
3. *The quotient G/H is Hausdorff if and only if H is closed.*
4. *If G is locally compact, then also G/H is locally compact.*
5. *If G is locally compact and $H \leq G$ is closed, for every compact set $C \subset G/H$ there exists a compact set $K \subset G$ such that $p(K) = C$.*

Proof 1. and 2. follow from the definitions and the properties of topological groups.

3. If G/H is Hausdorff, then points are closed. In particular $eH \in G/H$ is closed and hence $p^{-1}(eH) = H \leq G$ is closed.

Conversely let us suppose that H is closed and let xH and yH be distinct points in G/H. Then xHy^{-1} is a closed set not containing the identity in G. Thus $G \setminus xHy^{-1}$ is an open neighborhood of $e \in G$ and hence by Proposition 2.1 there exist U an open neighborhood of $e \in G$ such that $U^{-1}U \subset G \setminus xHy^{-1}$. Thus $U^{-1}U \cap xHy^{-1} = \varnothing$, that is UxH and UyH are disjoint neighborhood respectively of xH and yH.

4. We have to show that every point in G/H has a compact neighborhood. Let $p(x) \in G/H$ and, since G is locally compact, let $x \in U \subset C$ with U open and C compact. Then $p(U)$ is open (by 1.), $p(C)$ is compact (since p is continuous) and $p(x) \in p(U) \subset p(C)$.

5. Let U be an open relatively compact neighborhood of $e \in G$. Then $\{p(Ux)\}_{x\in G}$ is an open cover of C and hence there exists a finite subcover $C \subset \cup_{j=1}^n p(Ux_j)$. Then

$$K := \bigcup_{j=1}^n \overline{U} x_j \cap p^{-1}(C) \subset G\,.$$

is a compact subset in G such that $p(K) = C$. □

If G acts transitively on a space X, then there is an isomorphism of G-spaces $G/G_x \to X$, where $G_x = \mathrm{Stab}_G(x)$ for $x \in X$, given by the map $gG_x \mapsto gx$. If X is a topological space and the action of G on X is continuous, then the G-map is also continuous. If G is a locally compact second countable Hausdorff space and X is locally compact Hausdorff, then the bijection is a homeomorphism.

Example 2.27

1. Let us consider the action of $\mathrm{O}(n+1,\mathbb{R})$ on $S^n \subset \mathbb{R}^{n+1}$. Notice that $g \in \mathrm{O}(n+1,\mathbb{R})$ if and only if ${}^t g g = \mathrm{Id}$, which implies that $\|gv\| = \|v\|$ for all $v \in \mathbb{R}^{n+1}$; in particular S^n is preserved by $\mathrm{O}(n+1,\mathbb{R})$. Moreover this action is transitive, that is $\mathrm{O}(n+1,\mathbb{R})e_{n+1} = S^n$ and in fact even the $\mathrm{SO}(n,\mathbb{R})$-action is transitive on S^n. The stabilizer of $e_{n+1} \in S^n$ is
$$\mathrm{SO}(n+1,\mathbb{R})_{e_{n+1}} = \{g \in \mathrm{SO}(n+1,\mathbb{R}) : \, ge_{n+1} = e_{n+1}\} \simeq \left\{\begin{pmatrix} \mathrm{SO}(n,\mathbb{R}) & 0 \\ 0 & 1 \end{pmatrix}\right\} < \mathrm{SO}(n+1,\mathbb{R})\,,$$
so that
$$S^n \simeq \mathrm{SO}(n+1,\mathbb{R})/\,\mathrm{SO}(n,\mathbb{R})\,.$$

2. The upper half plane $H^2_{\mathbb{R}} := \{x + \imath y \in \mathbb{C} : \, y > 0\}$ is an $\mathrm{SL}(2,\mathbb{R})$-space, with the $\mathrm{SL}(2,\mathbb{R})$-action given by *fractional linear transformations*: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in H^2_{\mathbb{R}}$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}\,.$$

It is easy to see that the action is transitive since

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \imath = x + \imath y$$

and that $\mathrm{SL}(2,\mathbb{R})_\imath = \mathrm{SO}(2,\mathbb{R})$. Thus the map $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \to H^2_{\mathbb{R}}$ identifies the upper half plane as the $\mathrm{SL}(2,\mathbb{R})$-orbit of $\imath$.

3. The group $\mathrm{SL}(2,\mathbb{R})$ acts transitively also on $\mathbb{R} \cup \{\infty\}$ with $P = \mathrm{SL}(2,\mathbb{R})_\infty$, where P is as in (2.8).
4. We generalize now the action in (2). Let

$$\mathrm{Sym}_1^+(n) := \{X \in M_{n\times n}(\mathbb{R}) : X \text{ is symmetric, positive definite and } \det(X) = 1\} .$$

Then $\mathrm{SL}(n,\mathbb{R})$ acts transitively on $\mathrm{Sym}_1^+(n)$ via $gX = gXg^t$, for $g \in \mathrm{SL}(n,\mathbb{R})$ and $X \in \mathrm{Sym}_1^+(n)$. Moreover

$$\mathrm{SL}(n,\mathbb{R})_{Id_n} = \{g \in \mathrm{SL}(n,\mathbb{R}) : gId_ng^t = Id_n\} = \mathrm{SO}(n,\mathbb{R}) ,$$

so that

$$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \simeq \mathrm{Sym}_1^+(n) .$$

If $n = 2$ this is nothing but the example in (2) (Exercise).

5. We generalize now the example in (3). We consider

$$\mathbb{P}^{n-1}(\mathbb{R}) = \mathbb{P}(\mathbb{R}^n) := \{V \subset \mathbb{R}^n : \text{ is a subspace with } \dim V = 1\}$$

with the transitive $\mathrm{SL}(n,\mathbb{R})$-action. In this case

$$\mathrm{SL}(n,\mathbb{R})_{<e_1>} = \left\{ \begin{pmatrix} a & x \\ 0 & A \end{pmatrix} : a \in \mathbb{R}, a \neq 0, x \in \mathbb{R}^{n-1}, A \in \mathrm{GL}(n-1,\mathbb{R}), \det A = a^{-1} \right\}$$

and we identify $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{R})_{<e_1>}$ with $\mathbb{P}^{n-1}(\mathbb{R})$. If $n = 2$ this is the example in (3).

6. Let

$$L := \{\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_n : f_j \in \mathbb{R}^n, \text{ for } j = 1, \ldots, n, \det(f_1, \ldots, f_n) = 1\}$$

be the space of lattices of covolume one in $\mathbb{R}^n$.

The group $\mathrm{SL}(n,\mathbb{R})$ acts transitively on L via

$$g(\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_n) := \mathbb{Z}gf_1 + \cdots + \mathbb{Z}gf_n$$

and the stabilizer of $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ is $\mathrm{SL}(n,\mathbb{Z})$. Thus L can be identified with $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.

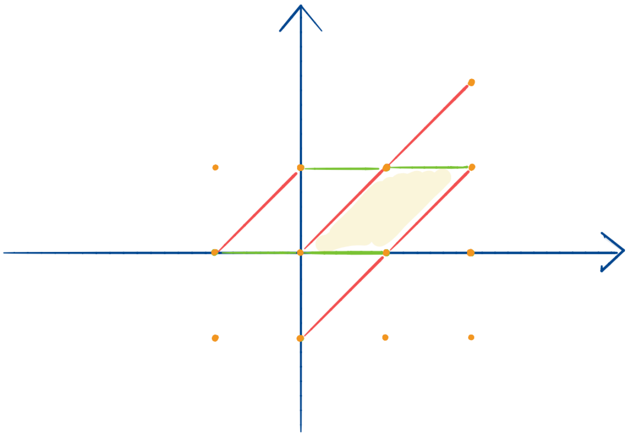


Figure 2.1: A lattice in L with $f_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $f_2 = \begin{pmatrix}1\\1\end{pmatrix}$

We prove now that if $H \trianglelefteq G$ is closed and normal, then $\Delta_G|_H = \Delta_H$. We start with the following lemma, only the first part of which (the definition) will be immediately used.

Lemma 2.6

Let G be a locally compact Hausdorff group and $H < G$ a closed subgroup. If $f \in C_c(G)$ and dh is the left Haar measure on H then

$$f^H(\dot{x}) := \int_H f(xh)\, dh$$

is in $C_c(G/H)$. Moreover the linear operator $A_H\colon C_c(G) \to C_c(G/H)$ defined as $A_H(f) := f^H$ is surjective. ♡

Proof The function f^H is obviously well defined as it is independent of the choice of representative of the coset xH. Moreover it is continuous[1]and $\operatorname{supp} f^H \subset p(\operatorname{supp} f)$. Thus $f^H \in C_c(G/H)$.

To prove the surjectivity, let $F \in C_c(G/H)$, let $C \subset G/H$ be the compact support of F and let $K \subset G$ be a compact set such that $p(K) = C$ (which exists by Proposition 2.5.5.). We will define $f \in C_c(G)$ such that $f^H = F$. Let $\eta \in C_c(G)$ such that $0 \leq \eta \leq 1$ and $\eta|_K \equiv 1$, which exists by Urysohn's Lemma ([7]). Then by definition

$$((F \circ p) \cdot \eta)^H = F \cdot \eta^H$$

[1]A function $f\colon G \to \mathbb{C}$ is right (resp. left) uniformly continuous if for every $\epsilon > 0$ there exist a neighborhood V of $e \in G$ such that $|f(s) - f(t)| < \epsilon$ for every $ts^{-1} \in V$ (resp. $t^{-1}s \in V$). Right uniform continuity follows from Lemma **??** applied to $X = G$ and the action of G on G by left translations: an analogous statement holds for left uniform continuity.

so that $F = \frac{((F\circ p)\cdot\eta)^H}{\eta^H}$. Thus we define

$$f(g) := \begin{cases} \frac{(F\circ p)(g)\cdot\eta(g)}{(\eta^H(p))(g)} & \text{if } (\eta^H(p))(g) \neq 0 \\ 0 & \text{if } (\eta^H \circ p)(g) = 0\,, \end{cases}$$

which we need to verify to be in $C_c(G)$. In fact obviously $\operatorname{supp} f \subset \operatorname{supp} \eta$. Moreover f is continuous as it is continuous on two open sets U_1 and U_2 whose union is G, namely on

1. $U_1 := \{g \in G : (\eta^H \circ))(g) \neq 0\}$ by definition and on
2. $U_2 := G \setminus KH$, where it vanishes. If fact if $g \in G \setminus KH$, then $p(g) \notin C = \operatorname{supp} F$, so that $(F \circ p)(g) = 0$.

So the only thing to verify is that $G = U_1 \cup U_2$. In fact, if $g \in G$ and $g \notin U_1$, then $0 = \eta^H \circ p(g)) = \int_H \eta(gh)\, dh$. Since $\eta \geq 0$ and η is continuous, this implies that $\eta(gh) = 0$ for all $h \in H$. Thus $gh \notin K$, which means that $g \notin KH$, that is $g \in U_2$.

Finally,

$$f^H(\dot{x}) = \int_H \frac{(F\circ p)(xh)\cdot\eta(xh)}{(\eta^H \circ p)(xh)}\, dh = \int_H F(\dot{x}) \frac{\eta(xh)}{\eta^H(xhH)}\, dh = F(\dot{x})\,.$$

□

Proof [Proof of Proposition 2.3] Since $H \trianglelefteq G$ and is closed, the quotient G/H is a locally compact Hausdorff topological group by Proposition 2.5.4. and hence there exists a left Haar measure on G/H, which we denote by $d\dot{x}$. We claim that the functional $m(f) := \int_{G/H} f^H(\dot{x})\, d\dot{x}$ is a left Haar measure on $C_c(G)$. In fact

$$m(\lambda(g)f) = \int_{G/H} (\lambda(g)f)^H(\dot{x})\, d\dot{x} \overset{(*)}{=} \int_{G/H} (\lambda(g)f^H)(\dot{x})\, d\dot{x} = \int_{G/H} f^H(\dot{x})\, d\dot{x} = m(f)\,,$$

where $(*)$ follows from the fact that left and right translations commute. If $t \in H$ and $f \in C_c(G)$), then

$$\begin{aligned} m(\rho(t)f) &= \int_{G/H} \left(\int_H (\rho(t)f)(xh)\, dh \right) d\dot{x} \overset{(**)}{=} \int_{G/H} \left(\int_H \Delta_H(t) f(xh)\, dh \right) d\dot{x} \\ &= \Delta_H(t) \int_{G/H} \left(\int_H f(xh)\, dh \right) d\dot{x} = \Delta_H(t) \int_{G/H} f^H(\dot{x})\, d\dot{x} = \Delta_H(t) m(f)\,, \end{aligned}$$

where we used in $(**)$ that dh is a left Haar measure. Comparing this with (2.7) shows that $\Delta_G|_H = \Delta_H$. □

We remark that the only place in the proof in which we used that H is a normal subgroup is to infer that there exists a left Haar measure on. G/H. The following result thus follows with the same proof.

Corollary 2.2

Let G be a locally compact Hausdorff topological group, $H \leq G$ a closed subgroup such that there exists a left invariant Borel measure on the topological space G/H. Then $\Delta_G|_H = \Delta_H$.

Luckily the above condition on the equality of the modular functions is not only necessary but also sufficient:

Theorem 2.4. (Weil Formula)

Let G be a locally compact Hausdorff topological group, $H \leq G$ a closed subgroup. Then there exists a left invariant positive Borel measure $d\dot{x}$ on G/H if and only if $\Delta_G|_H = \Delta_H$. Such an invariant measure is characterized by Weil formula

$$\int_G f(g)\, dg = \int_{G/H} \left(\int_H f(xh)\, dh \right) d\dot{x}$$

for every $f \in C_c(G)$, where dg and dh are the left Haar measures respectively on G and on H.

Corollary 2.3

Let G be a locally compact Hausdorff topological group, and let H' and H be closed subgroups with $H' \leq H \leq G$.

1. *If G and H are unimodular, there exists a unique (up to scalar) G-invariant measure on G/H.*
2. *If G, H and H' are unimodular, there exists invariant measures dx, dy and dz on G/H', on G/H and on H/H' such that*

$$\int_{G/H'} f(x)\, dx = \int_{G/H} \left(\int_{H/H'} f(yz)\, dz \right) dy$$

for all $f \in C_c(G/H')$.

Example 2.28 We look for an $\mathrm{SL}(2,\mathbb{R})$-invariant measure on the upper half plane $H^2_{\mathbb{R}} \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$. The group $\mathrm{SO}(2,\mathbb{R})$ is unimodular since it is compact, and $\mathrm{SL}(2,\mathbb{R})$ is also unimodular since it is equal to its own commutator subgroup,

$$\mathrm{SL}(2,\mathbb{R}) = [\mathrm{SL}(2,\mathbb{R}), \mathrm{SL}(2,\mathbb{R})] .$$

We will see later that this is true for all *semisimple Lie groups* (see Proposition 4.12, of which $\mathrm{SL}(2,\mathbb{R})$ is an example), but in this particular case we can see it from the fact that $\mathrm{SL}(2,\mathbb{R})$ can be generated by the upper triangular and the lower triangular matrices and that for $a, x \in \mathbb{R}$, $a \neq 0$,

$$\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & (a^2-1)x \\ 0 & 1 \end{pmatrix}.$$

Thus there exists a positive $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $H^2_{\mathbb{R}}$, which we proceed to compute.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z \in H^2_{\mathbb{R}}$ and $w := gz$, it is easy to check that

1. $\operatorname{Im} w = |cz+d|^{-2} \operatorname{Im} z$, and
2. $dw = (cz+d)^{-2} dz$,

so that $(\operatorname{Im} z)^{-2} dz\, d\bar{z}$ is an $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $H^2_{\mathbb{R}}$. Since $dz\, d\bar{z} = -2\imath\, dx\, dy$, $(\operatorname{Im} z)^{-2} |dz\, d\bar{z}| = y^{-2}\, dx\, dy$ is a positive $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $H^2_{\mathbb{R}}$. Thus Weil formula in this case reads

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(g)\, dg = \int_{H^2_{\mathbb{R}}} \left(\int_{\mathrm{SO}(2,\mathbb{R})} f(gk)\, dk \right) y^{-2}\, dx\, dy\,,$$

where we identify gk with $g\imath = x + \imath y$, and where $dk := \frac{1}{2\pi} d\theta$ is the normalized Haar measure on $\mathrm{SO}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$.

In this particular case, since also P acts transitively on $H^2_{\mathbb{R}}$, we can decompose the measure on $H^2_{\mathbb{R}}$ even further. In fact, the subgroup

$$P^+ := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

also acts transitively and freely on $H^2_{\mathbb{R}}$ (that is with trivial stabilizer). Moreover, just like $P = A \ltimes_\eta N$, we can also write $P^+ = A^+ \ltimes_\eta N$, where $A^+ \simeq (\mathbb{R}_{>0}, \cdot)$ is the subgroup of diagonal matrices with positive entries.

Thus, any $z = x + \imath y \in H^2_{\mathbb{R}}$ can be obtained by acting upon $\imath$ via the element $n(x)a(y) \in NA^+$ as follows:

$$\underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}_{=:n(x)} \underbrace{\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}}_{=:a(y)} \imath = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \imath y = x + \imath y\,.$$

If $\phi \in C_c(H^2_{\mathbb{R}})$, we can consider the composition

$$\begin{tikzcd} NA^+ \arrow[r] \arrow[rr, bend right, "\Phi"'] & H^2_{\mathbb{R}} \arrow[r, "\phi"] & \mathbb{R} \end{tikzcd}$$

namely

$$\phi(x+\imath y) = \phi(n(x)a(y)\imath) = \Phi(n(x)a(y))\,.$$

Thus

$$\int_{H^2_{\mathbb{R}}} \phi(x+\imath y)\,\frac{dx\,dy}{y^2} = \int_{\mathbb{R}_{>0}}\left(\int_{\mathbb{R}} \Phi(n(x)a(y))\,dx\right)\frac{dy}{y^2}\,,$$

so that we can write

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(g)\,dg = \int_{\mathbb{R}>0}\left(\int_{\mathbb{R}}\left(\int_{\mathrm{SO}(2,\mathbb{R})} f(n(x)a(y)k)\,dk\right)\,dx\right)\frac{dy}{y^2}\,. \tag{2.9}$$

Any element in P^+ (hence in $H^2_{\mathbb{R}}$) can be written also uniquely as the product of an element in A^+ and one in N. In fact an easy calculation shows that

$$n(x)a(y) = a(y)a(y)^{-1}n(x)a(y) = a(y)n(y^{-1}x)\,,$$

Thus with

$$\begin{tikzcd} A^+N \arrow[r] \arrow[rr, bend right, "\Psi"'] & H^2_{\mathbb{R}} \arrow[r, "\phi"] & \mathbb{R} \end{tikzcd}$$

we have

$$\begin{aligned}\int_{H^2_{\mathbb{R}}} \phi(x+\imath y)\,\frac{dx\,dy}{y^2} &= \int_{\mathbb{R}_{>0}}\left(\int_{\mathbb{R}} \Psi(a(y)n(y^{-1}x))\,dx\right)\frac{dy}{y^2}\\ &= \int_{\mathbb{R}_{>0}}\left(\int_{\mathbb{R}} y\Psi(a(y)n(x))\,dx\right)\frac{dy}{y^2}\\ &= \int_{\mathbb{R}_{>0}}\left(\int_{\mathbb{R}} \Psi(a(y)n(x))\,dx\right)\frac{dy}{y}\,.\end{aligned} \tag{2.10}$$

Thus

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(g)\,dg = \int_{\mathbb{R}>0}\left(\int_{\mathbb{R}}\left(\int_{\mathrm{SO}(2,\mathbb{R})} f(a(y)n(x)k)\,dk\right)\,dx\right)\frac{dy}{y}\,.$$

Notice that both (2.9) and (2.10) are examples of Weil's formulas. However in this case the group P^+ is not unimodular, while the subgroup N is. There is in fact the following more general version of Weil's formula that we state without proof.

Definition 2.6. (Quasi-invariant measure)

Let m be a measure on G/H. We say that m is quasi-invariant *if there exists a homomorphism $\chi\colon G \to \mathbb{R}_{>0}$ such that*

$$g_* m(A) = \chi(g) m(A)$$

for every $A \subset G/H$ measurable and every $g \in G$. The homomorphism χ is the modulus *of the quasi-invariant measure.*

Theorem 2.5. (Generalized Weil Formula)

Let G be a locally compact Hausdorff group and $H < G$ a closed subgroup. There exists a quasi-invariant measure on G/H with modulus χ if and only $\Delta_G|_H = \Delta_H \cdot \chi|_H$.

Example 2.29 On $S^1 \simeq \mathrm{SL}(2,\mathbb{R})/P$ there is no quasi-invariant measure since $\mathrm{SL}(2,\mathbb{R})$ is unimodular and P is not, so that such homomorphism χ does not exist. On the other hand, in the above example, since P^+ is not unimodular and N is, there could be a homomorphism χ that extends to P^+. The above calculation shows that this is indeed the case.

The following is a fundamental example to which one can apply the above discussion.

Definition 2.7. (Lattice Subgroup)

A lattice *Γ in a locally compact Hausdorff group G is a subgroup with the following properties:*

1. *Γ is discrete, and*
2. *there exists on G/Γ a finite G-invariant measure.*

Proposition 2.6

Let G be a topological group that admits a lattice $\Gamma < G$. Then G is unimodular.

Proof The modular function Δ_G is a homomorphism that contains Γ is its kernel. Hence it descends to a Δ_G-map $\Delta\colon G/\Gamma \to \mathbb{R}_{>0}$, that is a map such that for $h \in G$ and $x \in G/\Gamma$

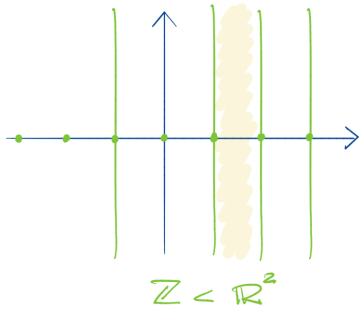
$$\Delta(hx) = \Delta_G(h)\Delta(x)\,.$$

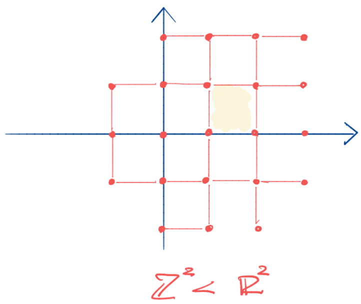
Thus the push-forward via Δ of the finite G-invariant measure on G/Γ is a finite $\Delta_G(G)$-invariant measure on $\mathbb{R}_{>0}$. This is impossible unless $\Delta_G(G) \equiv 1$.

Remark According to Proposition 2.6 a necessary condition for the existence of a lattice subgroup is that G is unimodular. Once this is verified, and $\Gamma < G$ is any discrete subgroup, Theorem 2.4 assures the existence of such a unique G-invariant measure on G/Γ. The point at stake here is thus the finiteness of the measure.

Example 2.30 We remarked already that the Lebesgue measure on $(\mathbb{R}^n, +)$ is the Haar measure and that $\mathbb{R}^n$ is unimodular.

1. The subgroup $\mathbb{Z} \simeq \{(n,0) : n \in \mathbb{Z}\} < \mathbb{R}^2$ is discrete but it is not a lattice because the fundamental domain of the action of $\mathbb{Z}$ on $\mathbb{R}^2$ is an infinite strip, which hence has infinite Lebesgue measure.
2. The subgroups $\mathbb{Z}^2 < \mathbb{R}^2$ and $\mathbb{Z}^n < \mathbb{R}^n$ are lattices whose covolume is easily computed.





Proof [Proof of Theorem 2.4] One direction has been proven already as Proposition 2.3 and Corollary 2.3.

To show the other direction, we make the following claim:

Claim 2.5.1. *Let $A_H : C_c(G) \to C_c(G/H)$ be the averaging operator defined in Lemma 2.6 and assume that $\Delta_G|_H = \Delta_H$. If $A_H(f_1) = A_H(f_2)$, then $\int_G f_1(g)\,dg = \int_G f_2(g)\,dg$.*

We assume that the claim is proven and proceed to conclude the proof. Let $F \in C_c(G/H)$ and let $f \in C_c(G)$ be such that $f^H = F$. The existence of such an f is assured by the surjectivity of A_H in Lemma 2.6. Because of the claim we can define a positive functional on $C_c(G/H)$ as follows:

$$\begin{aligned} m\colon C_c(G/H) &\longrightarrow \quad \mathbb{R} \\ F \qquad &\mapsto \int_G f(g)\,dg\,. \end{aligned}$$

By Riesz Representation Theorem, this is a positive Borel measure $d\dot{x}$ which is also left invariant because of the left invariance of dg. Then Weil formula follows at once:

$$\int_G f(g)\,dg = m(F) = \int_{G/H} F(\dot{x})\,d\dot{x} = \int_{G/H} \left(\int_H f(xh)\,dh \right) d\dot{x}\,.$$

To prove the claim it is enough that we show that, under those hypotheses, if $f^H = 0$, then $\int_G f(g)\,dg = 0$. This will follow immediately once we will have proven that, under the hypotheses of the Claim,

$$\int_G f_1(g) \left(\int_H f_2(gh)\,dh \right) dg = \int_G f_2(g) \left(\int_H f_1(gh)\,dh \right) dg\,. \tag{2.11}$$

In fact, if $f_2^H = 0$, then it follows from (2.11) that

$$0 = \int_G f_2(g) \left(\int_H f_1(gh)\,dh \right) dg = \int_G f_2(g) f_1^H(\dot{g})\,dg\,.$$

It is hence enough to find $f_1 \in C_c(G)$ such that $f_1^H \equiv 1$ on $\mathrm{supp}(f_2)$, which exists because of Lemma 2.6.

Thus we are left to prove (2.11), which is just a verification. In fact,

$$\begin{aligned}
\int_G f_1(g) \left(\int_H f_2(gh)\,dh \right) dg &\overset{(1)}{=} \int_H \left(\int_G f_1(g) f_2(gh)\,dg \right) dh \\
&\overset{(2)}{=} \int_H \left(\int_G f_1(gh^{-1}) f_2(g) \Delta_G(h)\,dg \right) dh \\
&\overset{(3)}{=} \int_G \left(\int_H f_1(gh^{-1}) \Delta_G(h)\,dh \right) f_2(g)\,dg \\
&\overset{(4)}{=} \int_G \left(\int_H f_1(gh) \underbrace{\Delta_H(h)^{-1} \Delta_G(h)}_{\equiv 1}\,dh \right) f_2(g)\,dg \\
&= \int_G f_2(g) \left(\int_H f_1(gh)\,dh \right) dg\,,
\end{aligned}$$

where we used

- in (1) and in (3) Fubini's theorem,
- in (2) the relation (2.7), and
- in (4) the second part of Proposition 2.2.

□

2.5.3 An Application

Theorem 2.6. (Mackey)

Let $\varphi\colon G\to H$ be a measurable homomorphism of locally compact second countable Haurdorff groups. Then φ is continuous.

Proof By replacing H with the closure of the image of φ, we may assume that the image of φ is dense in H. We want to show that for every open neighborhood $V\subset H$ of the identity $e_H\in H$, there exists an open neighborhood $N\subset G$ of the identity $e_g\in G$ such that $\varphi(N)\subset V$, that is $N\subset\varphi^{-1}(V)$.

Let $U\subset H$ be a symmetric open neighborhood of $e_H\in H$ such that $U^2\subset V$. Let $(h_n)\subset\varphi(G)$ be a countable dense set, which exists since H is second countable, and let $(g_n)\subset G$ be such that $\varphi(g_n)=h_n$. We can write $H=\cup_{n\in\mathbb{N}}h_nU$ and hence $G=\cup_{n\in\mathbb{N}}g_n\varphi^{-1}(U)$. If m is the left Haar measure on G, there exists $n\in\mathbb{N}$ such that $m(g_n\varphi^{-1}(U))>0$, so that $m(\varphi^{-1}(U))>0$. Since G is locally compact and m is inner regular, there exists a compact set $A\subset\varphi^{-1}(U)$ with $m(A)>0$. Then $\varphi^{-1}(V)\supset\varphi^{-1}(U)\varphi^{-1}(U)\supset AA^{-1}$, and it is hence enough to show that AA^{-1} contains an open neighborhood N of $e_g\in G$. □

Thus we need to prove the folllowing:

Lemma 2.7

Let G be a locally compact Hausdorff topological group. If $A\subset G$ is a compact set with $m(A)>0$, then $A^{-1}A$ contains an open neighborhood of $e_G\in G$.

Proof **[Perhaps fix left and right.]** If $Ax\cap A\neq\varnothing$, then $x\in A^{-1}A$, so that it is enough to show that

$$A^{-1}A=\{x:\ Ax\cap A\neq\varnothing\}\supset N\ni e_G\,,$$

where N is an open neighborhood of e_G. Since m is outer regular, then $m(A)=\inf\{m(W):\ W\supset A,\ W\text{ is open}\}$ and since $m(A)>0$, there exists an open set $W\supset A$ such that $m(W)<2\,m(A)$.

We will show that since A is compact, there exists an open neighborhood N of e_G such that $AN\subset W$, so that for every $x\in N$

$$\frac{1}{2}m(W)<m(A)=m(Ax)<m(W)\,.$$

This will be enough to conclude the proof, because in fact this open neighborhood N has the desired property that

$$e_G \in N \subset \{x :\ Ax \cap A \neq \varnothing\} = A^{-1}A\,.$$

In fact, if on the contrary for $x \in N$, $Ax \cap A = \varnothing$, then

$$m(Ax \cup A) = m(Ax) + m(A) = 2m(A) > m(W)\,.$$

But this is not possible since $Ax \subset W$ and $A \subset W$, imply that $Ax \cup A \subset W$ and hence $m(Ax \cup A) < m(W)$. [The idea is that whatever keeps A within W cannot translate A so that it is disjoint from itself.] □

Lemma 2.8

Let G be a topological group, $A \subset G$ a compact set and $W \subset G$ an open set such that $A \subset W$. Then there exists a neighborhood $N \ni e_G$ such that $AN \subset W$. ♡

Proof Since W is open, for all $x \in A$ there exists an open neighborhood $V_x \ni e_G$ such that $xV_x \subset W$. Let U_x be a symmetric open neighborhood of $e_G \in G$ such that $U_xU_x \subset V_x$. The sets $\{xU_x :\ x \in A\}$ form an open cover of A and, since A is compact, there exists a finite subcover $A \subset x_1U_{x_1} \cup \cdots \cup x_nU_{x_n}$. Let $N := U_{x_1} \cap \cdots \cap U_{x_n} \subset U_{x_j}$ for $j = 1, \ldots, n$. Then for $x_1, \ldots, x_n \in A$,

$$\begin{aligned} AN &\subset x_1U_{x_1}N \cup \cdots \cup x_nU_{x_n}N \\ &\subset x_1U_{x_1}U_{x_1} \cup \cdots \cup x_nU_{x_n}U_{x_n} \\ &\subset x_1V_{x_1} \cup \cdots \cup x_nV_{x_n} \subset W\,. \end{aligned}$$

□

Chapter 2 Exercise

1. Show that if X is a compact metric space, then $\mathrm{Iso}(X)$ is a closed subgroup of $\mathrm{Homeo}(X)$.
2. Verify that the Euclidean topology on $\mathrm{GL}(n, \mathbb{R})$ is the same as the compact-open topology.
3. Show that $\mathrm{Homeo}(S^2)$ is not locally compact.
4. Show that if X is a locally compact metric space, then $\mathrm{Iso}(X)$ is locally compact as well.
5. Show that $\mathrm{Aut}(\mathbb{R}^n, +) = \mathrm{GL}(n, \mathbb{R})$ and that $\mathrm{mod}_{\mathbb{R}^n} \colon \mathrm{GL}(n, \mathbb{R}) \to (\mathbb{R}_{>0}, \cdot)$ is $\mathrm{mod}_{\mathbb{R}^n}(\alpha) = |\det \alpha|$.
6. Let $\mathcal{H}$ be a Hilbert space over a field $k = \mathbb{R}$ or $\mathbb{C}$. Show that $\mathcal{H}$ is locally compact if and only if it is finite-dimensional.
7. Let X, Y, Z be topological space, and denote by $C(Y, X)$ the set of continuous maps from Y

to X. The set $C(Y,X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K,U) := \{f \in C(Y,X) \mid f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.
Prove the following useful facts about the compact-open topology.
If Y is locally compact, then:

(a). The evaluation map $e\colon C(Y,X) \times Y \to X, e(f,y) := f(y)$, is continuous.

(b). A map $f\colon Y \times Z \to X$ is continuous if and only if the map

$$\hat{f}\colon Z \to C(Y,X), \hat{f}(z)(y) = f(y,z),$$

is continuous.

8. (a). Let X be a compact Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.

(b). The objective of this exercise is to show that $(\mathrm{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.
Consider the "middle thirds" Cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \epsilon_n 3^{-n} : \epsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$

in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1-3^{-n}, 1]$.
Further we construct a sequence of homeomorphisms $h_n \in \mathrm{Homeo}(C)$ as follows:

- $h_n(x) = x$ for all $x \in C \setminus (U_n \cup V_n)$,
- $h_n(0) = 0$,
- $h_n(U_{n+1}) = U_n$,
- $h_n(U_n \setminus U_{n+1}) = V_{n+1}$,
- $h_n(V_n) = V_n \setminus V_{n+1}$.

These restrict to homeomorphisms $h_n|_X$ on $X := C \setminus \{0\}$.
Show that the sequence $(h_n|_X)_{n\in\mathbb{N}} \subset \mathrm{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n\in\mathbb{N}} \subset \mathrm{Homeo}(X)$ of their inverses does not!
Remark However, if X is locally compact and *locally connected* then $\mathrm{Homeo}(X)$ is a topological group.

(c). Let $\mathbb{S}^1 \subset \mathbb{C} \setminus \{0\}$ denote the circle. Show that $\mathrm{Homeo}(\mathbb{S}^1)$ is not locally compact.
Remark In fact, $\mathrm{Homeo}(M)$ is not locally compact for any manifold M of dimension at least one.

9. Let (X, d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology. *Hint:* Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem.

10. (a). The general linear group

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$\begin{aligned} j\colon\ \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Homeo}(\mathbb{R}^n), \\ A &\mapsto (x \mapsto Ax). \end{aligned}$$

(b). Show that $j(\mathrm{GL}(n, \mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n, \mathbb{R}^n)$ is endowed with the compact-open topology.

(c). If we identify $\mathrm{GL}(n, \mathbb{R})$ with its image $j(\mathrm{GL}(n, \mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$.

11. Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \pmod{p^n}\right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each $\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

(a). Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Z}_p, \\ x &\mapsto (x \pmod{p^n})_{n\in\mathbb{N}} \end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

(b). Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the "middle thirds" cantor set C as defined in Exercise 8. .

12. Let G be a topological group, X a topological space and $\mu : G \times X \to X$ a continuous transitive group action.

a) Show that if G is compact then X is compact.

b) Show that if G is connected then X is connected.

13. Let G be a connected topological group, $U \subset G$ an open neighborhood of the identity and $U^n := \{g_1 \cdots g_n \mid g_1, \ldots, g_n \in U\}$. Show that $G = \bigcup_{n=1}^{\infty} U^n$.
Hint: You may assume that $g^{-1} \in U$ for every $g \in U$. Why?

14. Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

15. (a). Let us consider the *three-dimensional Heisenberg group* $H = \mathbb{R} \rtimes_\eta \mathbb{R}^2$, where $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Verify that the Lebesgue measure is the Haar measure of $\mathbb{R} \rtimes_\eta \mathbb{R}^2$ and that the group is unimodular.

(b). Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Show that $\frac{da}{a^2}\, db$ is the left Haar measure and $da\, db$ is the right Haar measure. In particular, P is *not* unimodular.

16. Let G be a locally compact Hausdorff group. Show that if $H_1 \leq H_2 \leq G$ are closed subgroups and H_1, H_2, G are all unimodular then there exist invariant measures dx, dy, dz on $G/H_1, G/H_2$ and H_2/H_1 respectively such that

$$\int_{G/H_1} f(x)dx = \int_{G/H_2} \left(\int_{H_2/H_1} f(yz)dz \right) dy$$

for all $f \in C_c(G/H_1)$.

17. Let $G = \mathrm{SL}_2(\mathbb{R})$ and P be the subgroup of upper triangular matrices. Show directly that

there is no (non-trivial) finite G-invariant measure on G/P.

Hint: Identify $G/P \cong \mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ with the unit circle and consider a rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and a translation

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

18. Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \ldots, x_k \in D$ such that
    (a). $x_1, \ldots, x_k$ are linearly independent over $\mathbb{R}$, and
    (b). $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Chapter 3 Lie Groups

p. 48 (first paragraph of chapter 3.2) It says there that one could consider the set of smooth functions on U as a subset of the smooth functions on the whole manifold. (This is not true, I mean $1/x$ is smooth on $\mathbb{R} \setminus \{0\}$, and cannot be extended to a smooth function on $\mathbb{R}$.) I had the feeling that, what this paragraph wanted to explain was that every function on U can locally be seen as a function on M, that is the germs of the functions defined on the whole of M are equal to the germs of the functions defined on U.

3.1 Definitions and Examples

Definition 3.1. (Lie Group)

A Lie group *G is a group endowed with the structure of a smooth (finite dimensional) manifold with respect to which the group operations*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are smooth. ♣

Remark Lie groups are locally compact Hausdorff (since they are manifolds) and have a *dimension*, namely the dimension of the underlying manifold. We will only consider finite dimensional Lie groups.

Example 3.1 (See Example 2.2) $(\mathbb{R}^n, +)$ is a Lie group.

Example 3.2 (See Example 2.4) The matrix group $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. Note that for $n = 1$, this is the group $(\mathbb{R}^*, \cdot)$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Example 3.3 (See Example 2.21) The one-dimensional torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \cong \mathrm{SO}(2)$

is a Lie group.

Example 3.4 A finite direct product of Lie groups is a Lie group. It follows from the previous example that the n-dimensional torus $\mathbb{T}^n$ is a Lie group.

Example 3.5 Countable discrete groups are Lie groups, as they are 0-dimensional manifolds. We require them to be countable because we consider smooth manifolds to be second countable (see Appendix A.3).

Example 3.6 (See Example 2.10) An inverse limit of discrete groups in ngeneral is not a Lie group. Indeed, such a group is totally disconnected but not discrete, and thus its topology cannot be that of a manifold. In particular, profinite groups are not Lie groups.

Example 3.7 (See Example 2.6) If X is a topological space, we have seen in Example 2.16 that $\mathrm{Homeo}(X)$ is not necessarily locally compact, hence "too big" to be a Lie group.

Example 3.8 (See Example 2.7) If (X, d) is a locally compact metric space, we have seen in Example 2.17 that $\mathrm{Iso}(X)$ is a locally compact topological group. It may or may not be a Lie group. For example, if $(X, d) = (\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}})$, then $\mathrm{Iso}(X) \cong \mathbb{R}^n \rtimes \mathrm{O}(n, \mathbb{R})$. More generally, the Myers–Steenrod Theorem, [6], states that the isometry group of a Riemannian manifold is a Lie group.

Example 3.9 The groups $A_{\det}$ and N in Example 2.11 are Lie groups. The group $A_{\det}$ gets the Lie group structure from the identification $A_{\det} \simeq ((\mathbb{R}^*)^n, \cdot)$. On the other hand we can give N the structure of a smooth $\frac{n(n-1)}{2}$-manifold coming from the homeomorphism $N \simeq \mathbb{R}^{\frac{n(n-1)}{2}}$, but, as remarked in Example 2.11, this is not a (Lie) group homomorphism, unless $n \leq 2$.

Example 3.10 Let $V \subset \mathbb{R}^n$ be a k-dimensional linear subspace, $k < n$, and let $\{y_1, \dots, y_n\}$ be a basis of $\mathbb{R}^n$ such that $\{y_1, \dots, y_k\} \subset \{y_1, \dots, y_k, y_{k+1}, \dots y_n\}$ is a basis of V. With our choice of basis we have:

$$\begin{aligned} &\mathrm{Stab}_{\mathrm{GL}(n,\mathbb{R})}(V) \\ :=&\{g \in \mathrm{GL}(n, \mathbb{R}) : gV \subset V\} \\ =&\left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathrm{GL}(n, \mathbb{R}) : \; A \in \mathrm{GL}(k, \mathbb{R}), C \in \mathrm{GL}(n-k, \mathbb{R}), B \in M_{k \times (n-k)} \right\}. \end{aligned}$$

This is again a Lie group whose underlying manifold is diffeomorphic to $\mathrm{GL}(k, \mathbb{R}) \times \mathrm{GL}(n-k, \mathbb{R}) \times \mathbb{R}^{k \times (n-k)}$. As in the previous example, this diffeomorphism is not a group homomorphism.

Definition 3.2. Lie group homomorphism

A Lie group homomorphism *is a group homomorphism between Lie groups that is also smooth.* ♣

We will often write simply *homomorphism* if there is no risk of confusion.

It is easy to verify that all of the above examples are Lie groups. This is either because there is an easy identification, as manifolds, to another Lie group, or because any open subset of a smooth manifold is a smooth manifold with the induced structure. To treat other cases it will be useful to have the following:

Theorem 3.1. Regular submanifold

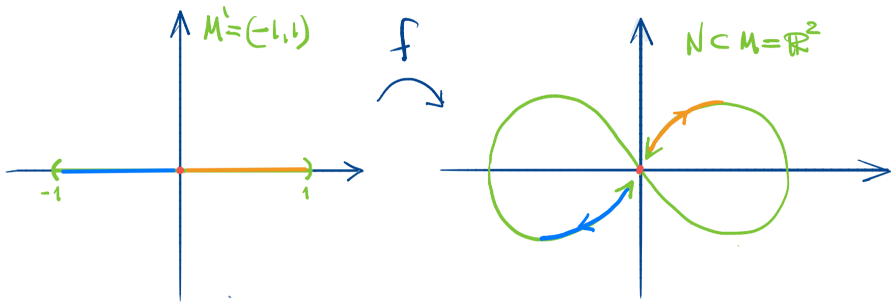
Let G be a Lie group and $H < G$ a subgroup that is also a regular submanifold. Then H is a Lie group with the induced smooth structure. ♡

The proof of the theorem boils down do the fact that restriction of a smooth map in $\mathbb{R}^n$ to a coordinate plane is again smooth.

Lemma 3.1

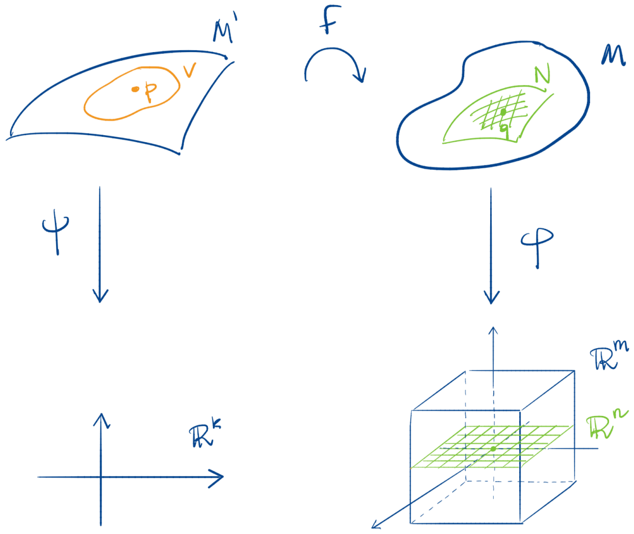
Let M, M' be smooth manifolds, $N \subset M$ a regular submanifold, and $f\colon M' \to M$ a smooth map such that $f(M') \subset N$. Then $f\colon M' \to N$ is also smooth. ♡

Remark [1, Remark 6.8] If N is not a regular submanifold then the lemma does not hold. For example consider the setting of the Example A.2 and consider a map $f\colon (-1,1) \to N$ such that $f(0) = 0$ that sends the interval $(0,1)$ into the right upper branch of the arc with the clockwise orientation and the interval $(-1,0)$ into the lower left branch with the counterclockwise orientation. Then f is smooth as a map into $M = \mathbb{R}^2$, but it is not even continuous if thought of as a map $f\colon (-1,1) \to N$ as the image of $[0, 1/2]$ is disconnected.



Then f is smooth as a function in $M = \mathbb{R}^2$ but not even continuous as a function in N.

Proof (Proof of Lemma 3.1)



Let $p \in M'$ and $f(p) =: q \in N$. If (U, φ) is a coordinate neighborhood around q as in Definition A.13, then if $m = \dim M > n = \dim N$, we have $\varphi(U) = (-\varepsilon, \varepsilon)^m, \varphi(q) = 0 \in \mathbb{R}^m$ and $U \cap N = \{y \in U : \varphi(y) = (y_1, \ldots, y_n, 0 \ldots, 0)\}$. If (V, ψ) is a coordinate neighborhood around p such that $f(V) \subset U$ and $x_1, \ldots, x_k$ are local coordinates in (V, ψ) for M', then the expression of $f\colon M' \to M$ in local coordinates is:

$$\varphi \circ f \circ \psi^{-1}(\underbrace{x_1, \ldots, x_k}_{x}) = (f_1(x), \ldots, f_n(x), 0, \ldots, 0).$$

However the expression of $f\colon M' \to N$ in local coordinates is

$$\varphi \circ f \circ \psi^{-1}(\underbrace{x_1, \ldots, x_k}_{x}) = (f_1(x), \ldots, f_n(x)),$$

that is the same expression followed by the projection $\mathbb{R}^{n+(m-n)} \to \mathbb{R}^n$. Since f is smooth, the proof is complete. $\square$

Proof (Proof of Theorem 3.1) Since $H < G$ is a regular submanifold, $H \times H \subset G \times G$ is also a regular submanifold., Therefore $m\colon H \times H \to G$ is a smooth map taking values in H, hence $m\colon H \times H \to H$ is also smooth, by Lemma 3.1. Similarly, inversion is smooth. $\square$

A very convenient way of determining whether a subset of a smooth manifold is a regular submanifolds is by using the Inverse Function Theorem:

Theorem 3.2. Inverse Function Theorem

Let M, M' be smooth manifolds of dimension m and k respectively, and let $f\colon M \to M'$ be a smooth map such that its rank is constant on M, say $\operatorname{rk} f = l$. Then for any $q \in f(M)$, $f^{-1}(q) \subset M$ is a closed regular submanifold of dimension $m - l$. ♡

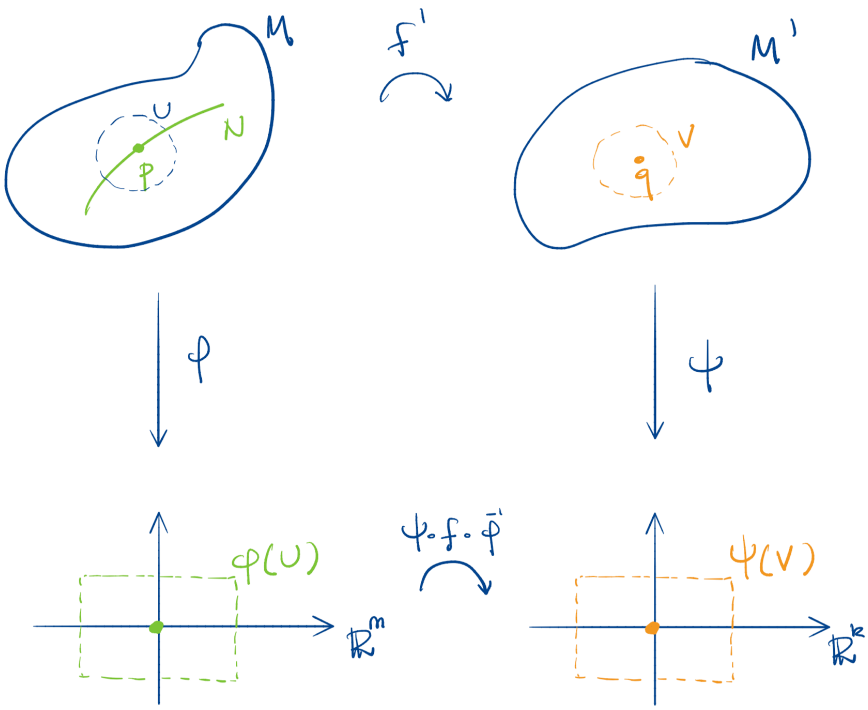
Remark Recall that the rank of a smooth map $f\colon M \to M'$ at a point $p \in M$ is

$$(\operatorname{rk} f)_p := \operatorname{rk} df_p = \dim(\operatorname{Im} df_p)\,.$$

If $f\colon M \to M'$ is a diffeomorphism, then the differential $d_p f$ at any point $p \in M$ is an isomorphism and hence $(\operatorname{rk} f)_p \equiv \dim M = \dim M'$. The rank assumption in the Inverse Function Theorem is essential: in fact for example any closed subset of $\mathbb{R}^n$ is the set of zeros of a smooth function $\mathbb{R}^n \to \mathbb{R}$.

The proof is a straightforward application of the rank condition, which gives the defining property of a regular submanifold.

Proof



Let $q \in f(M)$ and set $N := f^{-1}(q) \subset M$. Then N is closed. Now let $p \in N$. Since $\operatorname{rk} f$ is constant there exist coordinate neighborhoods (U, φ) for p and (V, ψ) for q such that:

1. $\varphi(p) = 0 \in \mathbb{R}^m$ and $\psi(q) = 0 \in \mathbb{R}^k$;
2. $\varphi(U) = (-\varepsilon, \varepsilon)^m$ and $\psi(V) = (-\varepsilon, \varepsilon)^k$;
3. $\psi \circ f \circ \varphi^{-1}(x_1, \ldots, x_m) = ((f_1(x), \ldots, f_l(x), 0, \ldots, 0)$.

This means that the points in U that map onto q (that is, the points of $U \cap N$) are exactly those whose first l local coordinates are 0. So

$$N \cap U = \varphi^{-1}(\psi \circ f \circ \varphi^{-1})^{-1}(0) = \varphi^{-1}(\{x \in (-\varepsilon, \varepsilon)^m : x_1 = \cdots = x_l = 0\}).$$

Thus N is a regular submanifold of M of dimension $m - l$. □

Example 3.11 We want to show that $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det A = 1\} = \det^{-1}(1)$ is a Lie group, where $\det\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ is the usual determinant map. Since $\mathrm{SL}(n, \mathbb{R})$ is a group and $\det$ is smooth, by Theorems 3.1 and 3.2 it suffices to check that the rank of $\det$ is constant. Given $X \in \mathrm{GL}(n, \mathbb{R})$, denote by $L_X\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ the left translation by X, $Y \mapsto XY$ and similarly, given $x \in \mathbb{R}^*$, by $l_x\colon \mathbb{R}^* \to \mathbb{R}^*$ the left translation by x, $t \mapsto xt$. Notice that they are both diffeomorphisms.

Now let $A \in \mathrm{GL}(n, \mathbb{R})$ and let $a := \det A \in \mathbb{R}^*$. The diagram

$$\begin{tikzcd} \mathrm{GL}(n,\mathbb{R}) \arrow[r, "\det"] \arrow[d, "L_A"'] & \mathbb{R}^* \arrow[d, "l_a"] \\ \mathrm{GL}(n,\mathbb{R}) \arrow[r, "\det"'] & \mathbb{R}^* \end{tikzcd}$$

is commutative, so that $\det = l_a \circ \det \circ L_{A^{-1}}$. By the chain rule, since $L_{A^{-1}}$ and l_a are diffeomorphisms, for every $X \in \mathrm{GL}(n, \mathbb{R})$

$$\mathrm{rk}_X \det = \mathrm{rk}_{A^{-1}X} \det$$

which is hence independent of A. By choosing $A = X$, we obtain $\mathrm{rk}\, d_X \det = \mathrm{rk}\, d_I \det$ for all $X \in GL(n, \mathbb{R})$. This shows that the rank is constant.

One can verify that $d_I \det = \mathrm{tr}$, the usual trace map $\mathbb{R}^{n \times n} \to \mathbb{R}$, and deduce that $\mathrm{SL}(n, \mathbb{R})$ is a Lie group of dimension $(n^2 - 1)$. (Exercise 2.).

Example 3.12 We consider now the orthogonal group $O(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : {}^t\!AA = I\}$, where ${}^t\!A$ is the transpose of A. If $f\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is defined by $f(X) = {}^t\!XX$, then $\mathrm{O}(n, \mathbb{R}) = f^{-1}(I)$, so that, using again Theorems 3.1 and 3.2, it suffices to check that the rank of

f is constant. We can show this in a way similar as for $\mathrm{SL}(n, \mathbb{R})$. In fact

$$f(XA^{-1}) = ({}^tA^{-1})({}^tX)XA^{-1} = L_{({}^tA^{-1})} \circ R_{A^{-1}} \circ f(X),$$

so that $f = L_{({}^tA^{-1})} \circ R_{A^{-1}} \circ f \circ R_A$. Just like before we can show that $\mathrm{rk}_X f = \mathrm{rk}_{XA} f$ is independent of A and hence, taking $A = X^{-1}$ it is constant and equal to $\mathrm{rk}_I f$.

Otherwise we could have computed the derivative directly:

$$\begin{aligned} d_X f(Y) &= \left.\frac{d}{ds}\right|_{s=0} {}^t(X + sY)(X + sY) = \\ &= \left.\frac{d}{ds}\right|_{s=0} ({}^tXX + s\,{}^tXY + s\,{}^tYX + s^2\,{}^tYY) = {}^tXY + {}^tYX. \end{aligned}$$

In particular

$$d_X f({}^tX^{-1}Z) = {}^tX({}^tX^{-1}Z) + {}^t({}^tX^{-1}Z)X = {}^tX{}^tX^{-1}Z + {}^tZX^{-1}X = Z + {}^tZ = d_I f(Z),$$

thus independent of X.

To compute the dimension of $O(n, \mathbb{R})$ we could use again Theorem 3.2, so that we need to compute $\mathrm{rk}_I f$. We just saw that

$$\begin{aligned} d_I f \colon \mathbb{R}^{n\times n} &\to \quad \mathbb{R}^{n\times n} \\ X \ &\mapsto X + {}^tX\,, \end{aligned}$$

that is $\mathrm{Im}\, d_I f$ consists of the symmetric matrices. Since a symmetric matrix is determined by its upper triangular part, the dimension of the image is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. As a result $\mathrm{O}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

We will later show the following powerful theorem:

Theorem 3.3. Closed Subgroup Theorem

Any closed subgroup of a Lie group is a Lie group. ♡

3.2 General Facts About Lie Groups and Lie Algebras

Let M be a smooth manifold. We denote by $C^\infty(M)$ the space of smooth $\mathbb{R}$-valued functions on M.

Definition 3.3. (Linear differential operator)

A linear operator $D\colon C^\infty(M) \to C^\infty(M)$ *is called a* linear differential operator *if:*

1. *For any open set* $U \subset M$*, and any smooth function on* M *with support in* U*, the support of* $D(f)$ *is in* U*, that is,* D *preserves the support of functions.*
2. *If* U *is an open set diffeomorphic to* $\mathbb{R}^n$*, then, under identification with* $\mathbb{R}^n$*,* D *takes on* U *the form of a usual differential operator. Namely if* $f \in C^\infty(M)$ *with support in* U*, then*

$$D(f) = \sum_{|\alpha|\leq k} g_\alpha D^\alpha f = \sum_{|\alpha|\leq k} g_\alpha \frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_n}},$$

where $g_\alpha \in C^\infty(M)$*,* $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$*,* $n = \dim M$ *and* $|\alpha| = \sum_{i=1}^{n} \alpha_i$*.*

The order *of a differential operator* D *is* $\operatorname{ord}(D) := \max\{|\alpha| : g_\alpha \neq 0\}$*. It can be verified that the order of a differential operator is independent of the choice of charts.* ♣

The space $\operatorname{DiffOp}(M)$ of linear differential operators on M is an algebra with composition as a product, which satisfies $\operatorname{ord}(D_1 D_2) \leq \operatorname{ord}(D_1) + \operatorname{ord}(D_2)$.

We are now going to look at a notable vector subspace of $\operatorname{DiffOp}(M)$, show that it is not a subalgebra and give it some weaker but essential structure. We will also see that one can do this abstractly.

We denote by $\operatorname{Vect}(M)$ the space of smooth vector fields on M (See Definition A.11).

If $U \subset M$ is open and $f \in C^\infty(M)$ with support in U, then Xf, defined as above, has also support in U. One can show that applying a vector field to a function at a point amounts to taking the derivative of that function in the direction of the vector field at that point. It follows that vector fields can be thought of as differential operators.

Proposition 3.1

There is a bijection between $\operatorname{Vect}(M)$ *and first order linear differential operators on* M *that vanish on constant functions.* ♠

Remark If $X, Y \in \operatorname{Vect}(M)$, then in general $\operatorname{ord}(XY) = 2$, so XY is not a vector field. Hence $\operatorname{Vect}(M)$ is a vector subspace of the algebra $\operatorname{DiffOp}(M)$, but not a subalgebra. On the other hand $XY - YX$ is a vector field, since it vanishes on constant functions and $\operatorname{ord}(XY - YX) = \operatorname{ord}(X) + \operatorname{ord}(Y) - 1 = 1$.

We want to give another structure to $\operatorname{Vect}(M)$, and we will view vector fields as derivations.

It is convenient now to look at the abstract setting of which this will be an example.

Definition 3.4. Derivation

Let $\mathbb{K}$ be a field, and let A be a $\mathbb{K}$-algebra (not necessarily associative). Let

$$\begin{aligned}\mathrm{End}(A) :=& \{\delta : A \to A \textit{ preserving the } \mathbb{K} - \textit{module structure}\} \\ =& \{\delta : A \to A : \ \delta(a+b) = \delta(a) + \delta(b) \textit{ and} \\ & \qquad \delta(\lambda a) = \lambda\delta(a) \textit{ for all } a, b \in A, \lambda \in \mathbb{K}\}\end{aligned}$$

An element $\delta \in End(A)$ is a derivation *of A if*

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in A$. We denote by $\mathrm{Der}(A)$ *the set of derivations of the $\mathbb{K}$-algebra A.* ♣

Example 3.13 The example to keep in mind is $A := C^\infty(M)$. In fact, the space of smooth functions on M form an $\mathbb{R}$-algebra, and vector fields are derivations of $C^\infty(M)$ because of the Leibniz rule.

We want to give the set of derivations some structure. Let $\delta_1, \delta_2 \in \mathrm{Der}(A)$. Then $\delta_1 + \delta_2 \in \mathrm{Der}(A)$ and $\lambda\delta_1 \in \mathrm{Der}(A)$ for all $\lambda \in \mathbb{R}$. Therefore $\mathrm{Der}(A)$ is a vector subspace of $\mathrm{End}(A)$. However it is not a subalgebra with the product defined as the composition, since $\delta_1\delta_2$ is not necessarily a derivation. In fact:

$$\begin{aligned}\delta_1\delta_2(ab) &= \delta_1(\delta_2(a)b + a\delta_2(b)) = \\ &= \delta_1\delta_2(a)b + \delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) + a\delta_1\delta_2(b).\end{aligned}$$

But then

$$\begin{aligned}(\delta_1\delta_2 - \delta_2\delta_1)(ab) &= \delta_1\delta_2(a)b + \cancel{\delta_2(a)\delta_1(b)} + \cancel{\delta_1(a)\delta_2(b)} + a\delta_1\delta_2(b) + \\ &\quad - \delta_2\delta_1(a)b - \cancel{\delta_1(a)\delta_2(b)} - \cancel{\delta_2(a)\delta_1(b)} - a\delta_2\delta_1(b) = \\ &= \delta_1\delta_2(a)b + a\delta_1\delta_2(b) - \delta_2\delta_1(a)b - a\delta_2\delta_1(b) = \\ &= (\delta_1\delta_2 - \delta_2\delta_1)(a)b + a(\delta_1\delta_2 - \delta_2\delta_1)(b).\end{aligned}$$

Hence $(\delta_1\delta_2 - \delta_2\delta_1) \in \mathrm{Der}(A)$. Let us introduce a notation for this: we define the *bracket* on

$\mathrm{Der}(A)$ as follows:

$$\begin{aligned}[\,\cdot\,,\,\cdot\,]\colon\ \mathrm{Der}(A)\times\mathrm{Der}(A) &\longrightarrow \qquad \mathrm{Der}(A)\\ (\delta_1,\delta_2) \quad &\mapsto [\delta_1,\delta_2] := \delta_1\delta_2 - \delta_1\delta_2\,.\end{aligned}$$

The following properties are then immediate to verify. If $\delta_1, \delta_2, \delta_3 \in \mathrm{Der}(A)$,

1. $[\cdot,\cdot]$ is bilinear.
2. $[\delta_1, \delta_2] = -[\delta_2, \delta_1]$.
3. $[\delta_1, [\delta_2, \delta_3]] = [[\delta_1, \delta_2], \delta_3] + [\delta_2, [\delta_1, \delta_3]]$.

Recalling that $\mathrm{Der}(A)$ is a only a vector space and not an algebra, we can extrapolate these properties and define a new algebraic structure on any vector space.

Definition 3.5. Lie algebra

A Lie algebra $\mathfrak{g}$ *over a field* $\mathbb{K}$ *is a* $\mathbb{K}$*-vector space with a bilinear map* $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ *satisfying the following properties for all* $X, Y \in \mathfrak{g}$*:*

1. $[X,Y] = -[Y,X]$.
2. $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$ *(Jacobi identity).*

♣

Remark The bracket is a sort of multiplication that is not associative. If it were, we would have $[X,[Y,Z]] = [[X,Y],Z]$, instead of the Jacobi identity.

Notice that the Jacobi identity is nothing but the defining relation of the derivation $\delta_X : \mathfrak{g}\to\mathfrak{g}$, defined by

$$\delta_X(Y) := [X,Y]\,.$$

where now we are thinking of a derivation of a Lie algebra (hence with respect to the Lie algebra operation $[\,\cdot\,,\,\cdot\,]$) rather than of the derivation of an algebra (see § 4.1).

Example 3.14

1. Any associative algebra is a Lie algebra with $[a,b] := ab - ba$.
2. $\mathrm{Vect}(M)$ is a Lie algebra with $[X,Y]_m(f) = X_m(Yf) - Y_m(Xf)$.
3. Any vector space V is a Lie algebra with the bracket $[v,w] = 0$ for all $v, v \in V$.

   Definition 3.6. Abelian Lie algebra

   A Lie algebra $\mathfrak{g}$ *is called* Abelian *if* $[X,Y] = 0$ *for all* $X, Y \in \mathfrak{g}$.

   ♣

4. The vector space $\mathbb{R}^{n\times n}$ of real $n\times n$ matrices is a Lie algebra with $[A,B] = AB - BA$

(this is a special case of the first item).

5. Let V be a two-dimensional vector space with basis $\{v, w\}$. Then we can define a bracket on the elements of the basis by $[v, v] = [w, w] = 0, [v, w] = -[w, v] = w$ and extend it by linearity to obtain a Lie algebra.
6. $\mathbb{R}^3$ with the cross product is a Lie algebra.
7. Let V be a three-dimensional vector space with basis $\{u, v, w\}$. Define a bracket on the elements of the basis by:

$$[u, u] = [v, v] = [w, w] = 0, [u, v] = w, [u, w] = -2u, [v, w] = 2v;$$

and extend it by linearity to obtain a Lie algebra. A matrix realization of this Lie algebra can be obtained by setting

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Related to the notion of Lie algebra there is also the notion of Lie subalgebra.

Definition 3.7. Lie subalgebra

Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra *if $[X, Y] \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$.* ♣

Remark Just like the concept of Lie algebra is weaker than the concept of algebra, the concept of Lie subalgebra is weaker than the concept of subalgebra.

Example 3.15

1. $\mathrm{DiffOp}(M)$ is an associative algebra (hence a Lie algebra) but we saw that $\mathrm{Vect}(M)$ is only a Lie subalgebra and not a subalgebra:

$$\underbrace{\mathrm{Vect}(M)}_{\text{Lie subalgebra}} \subset \underbrace{\mathrm{DiffOp}(M)}_{\text{subalgebra}} \subset \underbrace{\mathrm{End}(C^\infty(M))}_{\text{algebra}}.$$

2. In general $\mathrm{Der}(A) \subset \mathrm{End}(A)$ is a Lie subalgebra but not a subalgebra.

Definition 3.8. Lie algebra homomorphism

Let $\mathfrak{g}, \mathfrak{h}$ *be Lie algebras. A* Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ *is a linear map such that*

$$[\varphi(X), \varphi(Y)]_{\mathfrak{h}} = \varphi([X, Y]_{\mathfrak{g}})$$

for all $X, Y \in \mathfrak{g}$. ♣

Example 3.16 Let $f : M \to M'$ be a diffeomorphism. Define $f_*\colon \operatorname{Vect}(M) \to \operatorname{Vect}(M')$ by

$$(f_*X)_{m'} = d_{f^{-1}(m')} f X_{f^{-1}(m')}. \tag{3.1}$$

In other words, f_*X is the only vector field that makes the following diagram commute:

$$\begin{tikzcd} TM \arrow[r, "df"] & TM' \\ M \arrow[u, "X"] \arrow[r, "f"] & M' \arrow[u, "f_*X"'] \end{tikzcd}$$

Then it can be checked that f_* is a Lie algebra homomorphism, that is $f_*([X, Y]) = [f_*(X), f_*(Y)]$ for all $X, Y \in \operatorname{Vect}(M)$. See the Remark after Corollary 3.3 for a further discussion.

3.3 Invariant Vector Fields: the Lie Algebra of a Lie Group

Definition 3.9. Smooth action

A smooth action *of a Lie group* G *on a smooth manifold* M *by diffeomorphisms is a group homomorphism* $G \to \operatorname{Diffeo}(M)$ *such that the map*

$$G \times M \to M$$

$$(g, m) \mapsto gm$$

is smooth. ♣

If G acts on M, then there is an induced action on $\operatorname{Vect}(M)$ and hence a homomorphism $G \to \mathcal{L}(\operatorname{Vect}(M)) : g \mapsto g_*$ to the space of linear operators of the vector space $\operatorname{Vect}(M)$, where g_* is defined as for a general diffeomorphism in (3.1), namely $(g_*X)_m = d_{g^{-1}m} g X_{g^{-1}m}$.

Definition 3.10. Invariant vector field

We say that the vector field X is invariant *if $g_*X = X$ for all $g \in G$. We denote by $\mathrm{Vect}(M)^G$ the space of invariant vector fields.* ♣

We saw in Example 3.16 that if $f \in \mathrm{Diffeo}(M)$, then $f_* \in \mathcal{L}(\mathrm{Vect}(M))$ is a Lie algebra homomorphism. Thus, for $g \in G$, g_* is also a Lie algebra homomorphism; that is, $g_*([X,Y]) = [g_*X, g_*Y]$. If moreover $X, Y \in \mathrm{Vect}(M)^G$, then

$$g_*([X,Y]) = [g_*X, g_*Y] = [X,Y];$$

and so $[X,Y] \in \mathrm{Vect}(M)^G$. Thus we have proven the following:

Lemma 3.2

Invariant vector fields form a Lie subalgebra of $\mathrm{Vect}(M)$. ♡

We want to understand better $\mathrm{Vect}(M)^G$.

Proposition 3.2

If G acts transitively on M and $m_0 \in M$, the evaluation map $E_{m_0}\colon \mathrm{Vect}(M)^G \to T_{m_0}M$ is injective and linear. Thus $\mathrm{Vect}(M)^G$ is identified with a linear subspace of $T_{m_0}M$ and is hence finite-dimensional. ♠

Remark We recall again that G acts transitively on M if for all $m, m' \in M$ there exists $g \in G$ such that $gm = m'$. For example if $M = G$, then G acts transitively and freely on itself, that is the stabilizers of the action are trivial. However, if the element $g \in G$ such that $gm = m'$ is not unique, then there are non-trivial stabilizers $G_m = \{g \in G : gm = m\} \neq \{e\}$.

Proof The evaluation map E_{m_0}, $X \mapsto E_{m_0}(X) := X_{m_0} \in T_{m_0}M$ is clearly linear. It is defined on all of $\mathrm{Vect}(M)$, but it is injective only on $\mathrm{Vect}(M)^G$. In fact, let $X, Y \in \mathrm{Vect}(M)^G$ be such that $X_{m_0} = Y_{m_0}$. Let $m \in M$, and let $g \in G$ be such that $gm_0 = m$. Then:

$$X_m = (g_*X)_m = d_{g^{-1}m}gX_{g^{-1}m} = d_{m_0}gX_{m_0};$$
$$Y_m = (g_*Y)_m = d_{g^{-1}m}gY_{g^{-1}m} = d_{m_0}gY_{m_0}.$$

Since $X_{m_0} = Y_{m_0}$, it follows that $X_m = Y_m$. Since m was arbitrary, $X = Y$. □

Corollary 3.1

If G acts transitively on M, then $\mathrm{Vect}(M)^G$ is a finite-dimensional Lie algebra. ♡

An important case of transitive action is the one above, where $M = G$.

Definition 3.11. Lie algebra of a Lie group

Let G be a Lie group. Then the Lie algebra of G*, denoted by* $\mathfrak{g}$ *(or* $\mathrm{L}(G)$ *or* $\mathrm{Lie}(G)$*) is the Lie algebra of left invariant vector fields on* G*.* ♣

We want to identify $\mathrm{Vect}(G)^G$ better. So far we know that the evaluation map $E_e : \mathrm{Vect}(G)^G \to T_eG$ is injective, where $e \in G$ is the identity.

Proposition 3.3

The evaluation map $E_e : \mathrm{Vect}(G)^G \to T_eG$ *is bijective.* ♠

We shall see in § 3.4 how much structure this bijection preserves.

Proof In view of Proposition 3.2 we only need to show that E is surjective. Given $A \in T_eG$, we extend A to the whole group by invariance. Namely, we define $\tilde{A} \in \mathrm{Vect}(G)^G$ by

$$\tilde{A}_g := ((L_g)_*A)_g = d_{L_g^{-1}g}L_g(A_{L_g^{-1}g}) = d_eL_g(A);$$

where $L_g : G \to G$ is the left translation diffeomorphism on G. □

For a general transitive action of G on a manifold M, it is not true in general that there is a unique element g such that $gm_0 = m$. Then the statement of the previous proposition needs to be modified a bit.

We saw that associated to an action of G on M there is an action of G on $\mathrm{Vect}(M)$. Now we see that there is also another action, the *isotropy representation*. If $m_0 \in M$ and G_{m_0} is the stabilizer in G of m_0, then G_{m_0} acts on $T_{m_0}M$. In fact, if $g \in G_{m_0}$, then $d_{m_0}g : T_{m_0}M \to T_{gm_0}M = T_{m_0}M$ is a bijective linear map, so that it defines a homomorphism:

$$\rho : G_{m_0} \to \mathrm{GL}(T_{m_0}M) : g \mapsto d_{m_0}g.$$

Denote by $T_{m_0}M^{\rho(G_{m_0})}$ the G_{m_0}-invariant vectors in $T_{m_0}M$, that is the vectors $v \in T_{m_0}M$ such that $\rho(g)v = d_{m_0}gv = v$ for all $g \in G_{m_0}$.

Proposition 3.4

If G acts transitively on M, there is an identification $E_{m_0} : \mathrm{Vect}(M)^G \to T_{m_0}M^{\rho(G_{m_0})}$. ♠

Proof By definition, if $X \in \mathrm{Vect}(M)^G$, then $E_{m_0}(X) = X_{m_0} \in T_{m_0}M^{\rho(G_{m_0})}$. To find the inverse, let $v \in T_{m_0}M^{\rho(G_{m_0})}$ and, given $m \in M$, let $g \in G$ be such that $gm_0 = m$. Define then $X_m := d_{m_0}gv \in T_mM$. To see that this is well-defined, let $g' \in G$ be such that $g'm_0 = m$, so that $g^{-1}g' \in G_{m_0}$. We need to see that $d_{m_0}gv = d_{m_0}g'v$, that is that $v = (d_{m_0}g)^{-1}(d_{m_0}g')v = d_{m_0}(g^{-1}g')v$, which holds true since $g^{-1}g' \in G_{m_0}$ and

$v \in T_{m_0} M^{\rho(G_{m_0})}$. □

Example 3.17 We saw in Example 2.27 that $\mathrm{SO}(3,\mathbb{R})$ acts transitively on S^2, $\mathrm{SO}(3,\mathbb{R})e_3 = S^2$. Moreover the stabilizer of e_3 can be identified with $\mathrm{SO}(2,\mathbb{R})$. Thus $\mathrm{Vect}(S^2)^{\mathrm{SO}(3,\mathbb{R})} \cong (T_{e_3}S^2)^{\rho(\mathrm{SO}(2,\mathbb{R}))}$. However, identifying the tangent plane at e_3 with the xy-plane, one sees that the action of $\mathrm{SO}(2,\mathbb{R})$ is by rotations. Thus the only tangent vector at e_3 that is invariant under the isotropy representation is the zero vector, hence there are no $\mathrm{SO}(3,\mathbb{R})$-invariant vector fields on S^2, and no $\mathrm{O}(3,\mathbb{R})$-invariant vector fields on S^2.

3.4 Characterization of the Lie Algebra of a Lie Group

Let G be a Lie group. We defined its Lie algebra $\mathfrak{g}$ as $\mathrm{Vect}(G)^G$ with the bracket operation. Since $\mathrm{Vect}(G)^G \cong T_eG$ by Proposition 3.3, we want to endow T_eG with a bracket operation so that this vector space isomorphism is a Lie algebra isomorphism.

If $G = \mathrm{GL}(n,\mathbb{R})$, then tangent space $T_I \,\mathrm{GL}(n,\mathbb{R})$ is isomorphic as a vector space to $\mathbb{R}^{n\times n}$. However $\mathbb{R}^{n\times n}$ is an algebra with the usual bracket $[A,B] = AB - BA$ (Example 3.14), which thus induces a bracket on T_eG.

Proposition 3.5

The vector space isomorphism $\mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n,\mathbb{R})) \to \mathrm{Vect}(\mathrm{GL}(n,\mathbb{R}))^{\mathrm{GL}(n,\mathbb{R})}$ is a Lie algebra isomorphism, where the bracket in $T_I \,\mathrm{GL}(n,\mathbb{R})$ is the one coming from the algebra structure on $\mathbb{R}^{n\times n}$. ♠

Corollary 3.2

The Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $\mathrm{GL}(n,\mathbb{R})$ is isomorphic to $\mathbb{R}^{n\times n}$ with the usual bracket of matrices. ♡

Proof If $A \in \mathbb{R}^{n\times n}$, let $A_I \in T_I \,\mathrm{GL}(n,\mathbb{R})$ be the matrix A thought of as a tangent vector at $I \in \mathrm{GL}(n,\mathbb{R})$, and let $\tilde{A} \in \mathrm{Vect}(\mathrm{GL}(n,\mathbb{R}))^{\mathrm{GL}(n,\mathbb{R})}$ be the invariant vector field obtained by spreading A_I around using invariance:

$$\begin{array}{ccccc} \mathbb{R}^{n\times n} & \cong & T_I \,\mathrm{GL}(n,\mathbb{R}) & \xrightarrow{f} & \mathrm{Vect}(\mathrm{GL}(n,\mathbb{R}))^{\mathrm{GL}(n,\mathbb{R})} \\ A & \leftrightarrow & A_I & \mapsto & \tilde{A} \end{array}.$$

We want to show that $f([A,B]) = [f(A), f(B)]$, that is:

$$\widetilde{[A,B]} = [\tilde{A}, \tilde{B}] \tag{3.2}$$

where $[A, B]$ is the bracket in $\mathbb{R}^{n\times n}$ and $[\tilde{A}, \tilde{B}]$ is the bracket in $\mathrm{Vect}(\mathrm{GL}(n, \mathbb{R}))^{\mathrm{GL}(n,\mathbb{R})}$.

First reduction. The equality (3.2) must hold everywhere on $\mathrm{GL}(n, \mathbb{R})$. However $\mathrm{Vect}(\mathrm{GL}(n, \mathbb{R}))^{\mathrm{GL}(n,\mathbb{R})}$ is a Lie algebra, so that the bracket of two invariant vectors is invariant. Thus $\widetilde{[A, B]}$ and $[\tilde{A}, \tilde{B}]$ are both invariant vector fields, and hence it suffices to prove the assertion at the identity, that is:

$$(\widetilde{[A, B]})_I = ([\tilde{A}, \tilde{B}])_I\,. \tag{3.3}$$

Second reduction. To show the equality (3.3) in $T_I\,\mathrm{GL}(n, \mathbb{R})$ it is enough to show that if $\lambda\colon T_I\,\mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ is a linear functional, since linear functionals separate points, then $\lambda((\widetilde{[A, B]})_I) = \lambda(([\tilde{A}, \tilde{B}])_I)$. By our correspondence, if $X \in T_I\,\mathrm{GL}(n, \mathbb{R})$, then $(\tilde{X})_I = X$, so we need to show that $\lambda([A, B]) = \lambda([\tilde{A}, \tilde{B}]_I)$. But $[A, B]$ is the bracket in $\mathbb{R}^{n\times n}$, hence it is enough to show:

$$\lambda([\tilde{A}, \tilde{B}]_I) = \lambda(AB) - \lambda(BA)\,. \tag{3.4}$$

Remark If $L\colon \mathbb{R}^m \to \mathbb{R}^k$ is a linear map, then its derivative $dL\colon T\mathbb{R}^m \to T\mathbb{R}^k$ is such that for every $x \in \mathbb{R}^m$, $d_xL\colon T_x\mathbb{R}^m \to T_{L(x)}\mathbb{R}^k$ can be identified with $d_xL\colon \mathbb{R}^m \to \mathbb{R}^k$ and in fact with $L\colon \mathbb{R}^m \to \mathbb{R}^k$. In other words the map

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathrm{End}(\mathbb{R}^m, \mathbb{R}^k) \\ x & \mapsto & d_xL \end{array}$$

is constant and identically equal to L.

We can hence think of L as its own derivative.

Third reduction. Given $\lambda\colon T_I\,\mathrm{GL}(n, \mathbb{R}) = \mathbb{R}^{n\times n} \to \mathbb{R}$, then

$$\lambda([\tilde{A}, \tilde{B}]_I) = d_I\lambda([\tilde{A}, \tilde{B}]_I) = [\tilde{A}, \tilde{B}]_I(\lambda) = \tilde{A}_I(\tilde{B}(\lambda))(I) - \tilde{B}_I(\tilde{A}(\lambda))(I).$$

By putting this together with (3.4), we are left to show that

$$\lambda(AB) - \lambda(BA) = \tilde{A}(\tilde{B}(\lambda))(I) - \tilde{B}(\tilde{A}(\lambda))(I).$$

We will show that $\lambda(AB) = \tilde{A}_I(\tilde{B}(\lambda))$, which concludes the proof.

$$\begin{aligned}\tilde{A}_I(\tilde{B}(\lambda)) = \tilde{A}_I(g \mapsto \tilde{B}_g(\lambda)) &\overset{(1)}{=} \tilde{A}_I(g \mapsto (d_IL_g)B_I(\lambda)) \overset{(2)}{=} \tilde{A}_I(g \mapsto (d_g\lambda)(d_IL_g)B_I) \\ &= \tilde{A}_I(g \mapsto d_I(\lambda \circ L_g)B_I) \overset{(3)}{=} \tilde{A}_I(g \mapsto (\lambda \circ L_g)(B)) = \tilde{A}_I(g \mapsto \lambda(gB)) \overset{(4)}{=} \lambda(AB)\,,\end{aligned}$$

□

where we used

– in (1) that $\tilde{B}$ is invariant;
– in (2) the interpretation in (A.1) of the action of the vector field $(d_I L_g)B_I$ on the function λ;
– in (3) the above remark, and
– in (4) the fact that the map $F\colon g \mapsto \lambda(gB)$ is linear in g for a fixed B and hence, again using the above remark and since $\tilde{A}_I = a$,, $\tilde{A}_I(F) = d_I F(A_I) = F(A_I) = \lambda(AB)$.

We want to find now a way of identifying the Lie algebra structure of a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. We want to see that, in fact, even for a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, we can use the matrix bracket.

Proposition 3.6

If $\varphi\colon G \to H$ is a Lie group homomorphism, then $d_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. ♠

Corollary 3.3

If $G \leq H$ is an inclusion of Lie groups, then $T_e G \hookrightarrow T_e H$ is an inclusion of tangent spaces that defines a Lie algebra embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$. ♡

It follows that if $H \leq \mathrm{GL}(n, \mathbb{R})$ is a Lie group, then the bracket on $\mathfrak{h}$ is the one coming from $\mathfrak{gl}(n, \mathbb{R})$.

In order to prove the proposition we need some preliminary remarks:

Remark

1. The differential is a local definition in the following sense. Let M, M' be smooth manifolds and $\varphi\colon M \to M'$ a smooth map. If $X \in \mathrm{Vect}(M)$, then in general $d\varphi(X)$ does not define a vector field on M'. In fact $\varphi(M)$ might not be the whole of M'. However even if it were, $d_p\varphi(X_p)$ is a tangent vector at $\varphi(p)$ that is however not necessarily uniquely defined, if for example there exists $q \in M$ such that $\varphi(p) = \varphi(q)$ but $d_p\varphi(X_p) \neq d_q\varphi(X_q)$. We say that $X \in \mathrm{Vect}(M)$ and $X' \in \mathrm{Vect}(M')$ are *φ-related* if $X' \circ \varphi = d\varphi \circ X$, that is if $X'_{\varphi(m)} = d_m\varphi(X_m)$ for all $m \in M$:

$$\begin{tikzcd} TM \arrow[r, "d\varphi"] & TM' \\ M \arrow[u, "X"] \arrow[r, "\varphi"] & M' \arrow[u, "\varphi_* X"'] \end{tikzcd}$$

It is easy to verify that if $X_i \in \mathrm{Vect}(M)$ is φ-related to $X_i' \in \mathrm{Vect}(M')$, for $i = 1, 2$, then $[X_1, X_2]$ is φ-related to $[X_1', X_2']$. (Exercise 4.)

Of course, because of Proposition 3.6, the above remark does not apply to homomorphisms between Lie groups and to left invariant vector fields. In fact we record the following fact:

Lemma 3.3

Let G, H be Lie groups with Lie algebras $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(H) = \mathfrak{h}$. Let $\varphi\colon G \to H$ be a Lie group homomorphism. Then the left invariant vector fields defined by $X \in \mathfrak{g}$ and $d_e\varphi(X) \in \mathfrak{h}$ are φ-related for all $X \in \mathfrak{g}$. ♡

So the property of being φ-related is in some sense a generalization of the property of being left invariant, in the sense of the following remark.

2. A vector field $X \in \mathrm{Vect}(G)$ is left invariant if and only if it is L_g-related to itself. In fact,

$$X \text{ is } L_g\text{-related to itself} \iff X \circ L_g = dL_g \circ X \iff X \circ L_g(h) = dL_g(X)(h) \text{ for all } h \in G$$
$$\iff X_{gh} = d_h L_g X_h \text{ for all } h \in G \iff X \text{ is left invariant}\,.$$

3. If $\varphi : G \to H$ is a homomorphism, then for all $g, h \in G$ we have $\varphi(gh) = \varphi(g)\varphi(h) \Rightarrow \varphi(L_g h) = L_{\varphi(g)}\varphi(h)$ and so $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$ for all $g \in G$.

Proof [Proof of Proposition 3.6]) We use the notation $\overline{X} := d\varphi(X)$. It will then be enough to show that X and $\overline{X}$ are φ-related. In fact, by 1. we have that $[X, Y]$ and $[\overline{X}, \overline{Y}]$ are φ-related, that is

$$d\varphi \circ [X, Y] = [\overline{X}, \overline{Y}] \circ \varphi = [d\varphi(X), d\varphi(Y)] \circ \varphi\,.$$

This is true at every point $g \in G$ and in particular at $g = e_G$. Since $\varphi(e_G) = e_H$, the assertion follows. So we only have to show that X and $\overline{X}$ are φ-related, that is $\overline{X}_{\varphi(g)} = d_g\varphi(X)$ for all

$g \in G$. But

$$\begin{aligned}\overline{X}_{\varphi(g)} &\overset{(1)}{=} d_{e_H} L_{\varphi(g)} \overline{X}_{e_H} \overset{(2)}{=} d_{e_H} L_{\varphi(g)} d_{e_G} \varphi(X_{e_G}) \\ &= d_{e_G}(L_{\varphi(g)} \circ \varphi)(X_{e_G}) \overset{(3)}{=} d_{e_G}(\varphi \circ L_g)(X_{e_G}) \\ &= d_g \varphi(X)(g)\,,\end{aligned}$$

where we used in (1) the fact the invariance of $\overline{X}$, in (2) its definition and in (3) 3.. □

We can now compute some Lie algebras.

Example 3.18 $\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) : \det(g) = 1\}$. So if $\gamma\colon (-\varepsilon,\varepsilon) \to \mathrm{SL}(n,\mathbb{R})$ is a smooth curve, then $\det(\gamma(t)) \equiv 1$, and so $\frac{d}{dt}\det(\gamma(t)) \equiv 0$. If moreover γ is chosen so that $\gamma(0) = I$, and so $\gamma'(0) \in T_I\,\mathrm{SL}(n,\mathbb{R}) = \mathfrak{sl}(n,\mathbb{R})$, then by the chain rule and by Exercise 4.:

$$0 = \left.\frac{d}{dt}\right|_{t=0} \det\gamma(t) = d_{\gamma(0)}\det\gamma'(0) = d_I \det\gamma'(0) = \mathrm{tr}(\gamma'(0)).$$

Thus $\mathfrak{sl}(n,\mathbb{R}) \subseteq \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}(A) = 0\}$. But since $\dim\mathfrak{sl}(n,\mathbb{R}) = \dim\{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}(A) = 0\} = n^2 - 1$, then equality holds.

In many cases it will be useful to have the following:

Lemma 3.4

Let $A, B\colon (-\varepsilon,\varepsilon) \to \mathbb{R}^{n\times n}$ be smooth curves and let us define $\varphi(s) := A(s)B(s) \subset \mathbb{R}^{n\times n}$. Then

$$\varphi'(s) = A'(s)B(s) + A(s)B'(s).$$

♡

Proof [Hint of the proof] Write in coordinates and mimic the proof of the product rule for $\mathbb{R}$-valued functions, where one uses only that $\mathbb{R}$ is an algebra and not necessarily a commutative one. □

Example 3.19 $\mathrm{O}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) : {}^t g g = I\}$. If $\gamma\colon (-\varepsilon,\varepsilon) \to \mathrm{O}(n,\mathbb{R})$ is a smooth curve then ${}^t\gamma(s)\gamma(s) \equiv I$. If $\gamma(0) = I$, then

$$0 = \left.\frac{d}{ds}\right|_{s=0} ({}^t\gamma(s)\gamma(s)) = {}^t\gamma'(0)\gamma(0) + {}^t\gamma(0)\gamma'(0) = {}^t\gamma'(0) + \gamma'(0).$$

Thus $\mathrm{Lie}(\mathrm{O}(n,\mathbb{R})) = \mathfrak{o}(n,\mathbb{R}) \subseteq \{A \in \mathfrak{gl}(n,\mathbb{R}) : {}^tA + A = 0\}$ is the space of skew-symmetric matrices. As before, by checking the dimensions one sees that this is an equality.

Example 3.20 $\mathrm{O}(p,q) = \{g \in \mathrm{GL}(n,\mathbb{R}) : gJ^tg = J\}$, where $J = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$. Let $\gamma\colon (-\varepsilon,\varepsilon) \to \mathrm{O}(p,q)$ be a smooth curve with $\gamma(0) = I$. Then with the same procedure as above one concludes that $\mathrm{Lie}(\mathrm{O}(p,q)) = \mathfrak{o}(p,q) = \{A \in \mathfrak{gl}(p+q,\mathbb{R}) : AJ + J^tA = 0\}$. If we write matrices in block form we deduce that

$$\mathfrak{o}(p,q) = \left\{ \begin{pmatrix} \mathfrak{o}(p,\mathbb{R}) & A \\ {}^tA & \mathfrak{o}(q,\mathbb{R}) \end{pmatrix} : A \text{ is a } p \times q \text{ matrix} \right\}.$$

Example 3.21 $N_1 = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \right\}$, then $\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \right\}$.

Example 3.22 $A_{\det} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_i \neq 0 \right\}$, then $\mathfrak{a}_{\det} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R} \right\}$.

Example 3.23 Let $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) : A^t\overline{A} = I\}$ be the group of unitary matrices. Then $\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : A + {}^t\overline{A} = 0\}$, the group of skew-Hermitian matrices.

Example 3.24 If

$$\mathrm{Sp}(2n,\mathbb{C}) = \{A \in \mathrm{GL}(2n,\mathbb{C}) : {}^tAFA = F\},$$

where $F = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and $\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{GL}(2n,\mathbb{R})$, then

$$\mathfrak{sp}(2n,\mathbb{C}) = \{A \in \mathfrak{gl}(2n,\mathbb{C}) : {}^tAF + FA = 0\} \tag{3.5}$$

and $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{sp}(2n,\mathbb{C}) \cap \mathfrak{gl}(n,\mathbb{R})$.

We just proved that if $\varphi : G \to H$ is a Lie group homomorphism, then $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. In order to have a correspondence between Lie groups and Lie algebras that is as complete as possible, we would like to have the converse of this statement. However, given a Lie group G we have defined its Lie algebra $\mathfrak{g}$, but we have not said proven that given a Lie algebra $\mathfrak{g}$ there is a Lie group G that "integrates" $\mathfrak{g}$ (that is such that $Lie(G) = \mathfrak{g}$). So the converse of Proposition 3.6 entails two different questions:

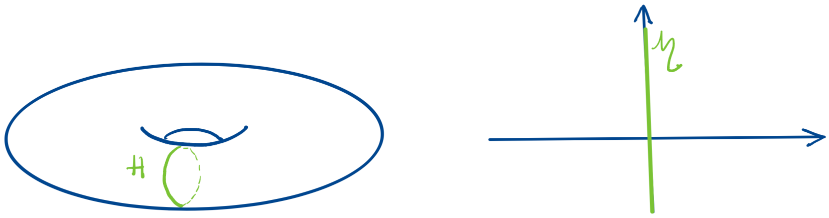
1. Given a Lie group G with Lie algebra $\mathfrak{g}$ and given a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, is there a subgroup $H \leq G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$?
2. If G, H are Lie groups and $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of their Lie algebras, does there exists a Lie group homomorphism $\varphi\colon G \to H$ such that $d_e\varphi = \pi$?

Definition 3.12

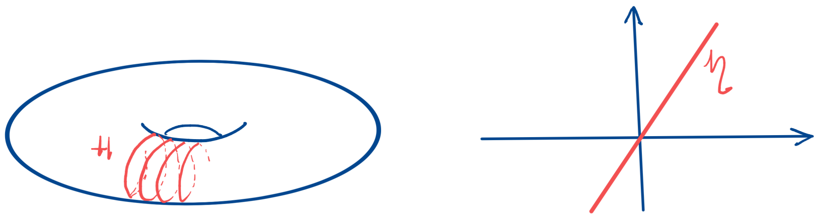
If $\mathfrak{g}, \mathfrak{h}$ are Lie algebras, the product $\mathfrak{g} \times \mathfrak{h}$ has the Lie algebra structure defined by $[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2], [Y_1, Y_2])$ for all $X_1, X_2 \in \mathfrak{g}$ and all $Y_1, Y_2 \in \mathfrak{h}$. ♣

Example 3.25 Let $G = \mathbb{T}^2 = S^1 \times S^1$. Since $\mathbb{R}^2 \to \mathbb{T}^2$ is a covering map, $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{T}_0\, \mathbb{T}^2 = \mathrm{T}_0\, \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

1. Let $\mathfrak{h} = \mathbb{R} \simeq \{0\} \times \mathbb{R}$. It is immediate that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and if $i\colon S^1 \to \mathbb{T}^2$ is defined as $i(S^1) := \{0\} \times S^1$, then $\mathrm{Lie}(i(S^1)) = \mathfrak{h}$. (In this case we do not even need Definition 3.12 as $\mathfrak{g}$ is Abelian and hence the bracket is trivial. See Corollary 3.6.)



2. Let $\mathfrak{h} = \{(x, y) \in \mathbb{R}^2 : y = \sqrt{2}x\}$. Let $\varphi\colon \mathbb{R} \to \mathbb{T}^2$ be defined as $\varphi(t) = (e^{it}, e^{i\sqrt{2}t})$. Then φ is an injective smooth homomorphism such that $\varphi(\mathbb{R}) =: H$ is a subgroup of $\mathbb{T}^2$ with Lie algebra $\mathfrak{h}$.



It is clear that we cannot expect that the subgroup H will be more than an immersed submanifold. In fact we have:

Theorem 3.4. Lie group - Lie algebra correspondence

If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, then there exists a unique immersed connected Lie subgroup $H < G$ with Lie algebra $\mathfrak{h}$. ♡

The proof will rely on Frobenius' Theorem, which we introduce with an example.

If M is a smooth manifold, $p \in M$ and $X \in \mathrm{Vect}(M)$, the theorem of existence and uniqueness of solutions of ODEs assures that there exists an $\epsilon > 0$ and a smooth curve $\gamma_p\colon (-\epsilon, \epsilon) \to M$ such that $\gamma_p(0) = p$ and $\gamma_p'(t) = X_{\gamma_p(t)}$ for all $t \in \mathbb{R}$. The curve γ_p is called *integral curve of* X, it is an immersed sumbanifold and has the property that its tangent space is spanned by X.

Suppose instead now to have two vector fields, $X_1, X_2 \in \mathrm{Vect}(M)$ and to look for a surface whose tangent space at every point is spanned by X_1 and X_2. Now this amounts to solving a linear system of PDEs and a solution in a neighborhood of a point $p \in M$ will be a surface whose tangent space at every point $q \in M$ "close enough" to p is spanned by X_1 and X_2. One possible way of finding such a surface is to consider the integral curve $\gamma_{1,p}$ of X_1, move along such a curve for a small amount of time, then move along the integral curve $\gamma_{2,p}$ of X_2. If such a surface exists, its tangent subspace at q will certainly contain X_2, but will have lost memory of X_1. Likewise, we could have started following first X_2 and then X_1 and now the tangent space to this hypothetical surface will contain X_1 and will have lost memory of X_2. It is clear that X_1 and X_2 must satisfy some relationship if we want the surface and its tangent space to be defined.

Definition 3.13. Distribution

1. *Let M be a manifold of dimension $n+k$ and for each $p \in M$ consider an n-dimensional subspace $\mathcal{D}_p \subset T_pM$. Suppose that in a neighborhood U of any point $p \in M$ there are n linearly independent smooth vector fields $X_1, \dots, X_n$ that give a basis of $\mathcal{D}_q$ for all $q \in U$. We then say that $\mathcal{D}$ is a* smooth distribution *of dimension n on M and that $X_1, \dots, X_n$ is a* local basis *of $\mathcal{D}$.*
2. *We say that a distribution is* involutive *if there exists a local basis $X_1, \dots, X_n$ of $\mathcal{D}$ such that $[X_i, X_j] \in \mathcal{D}$ for all $1 \le i, j \le n$.*
3. *If $\mathcal{D}$ is a smooth distribution and $\varphi\colon N \to M$ is a one-to-one immersion, we say that $\varphi(N)$ is an* integral submanifold *of $\mathcal{D}$ if $d_p\varphi T_pN \subset \mathcal{D}_{\varphi(p)}$.*
4. *We say that a distribution $\mathcal{D}$ on M is* completely integrable *if through each point in M there is an integral submanifold $\varphi : N \to M$ such that $d_p\varphi T_pN = \mathcal{D}_{\varphi(p)}$ for all $p \in N$.*

♣

From the above discussion it should be clear that finding conditions for the existence of integral submanifolds amounts to finding conditions for the existence of solutions to linear systems of PDEs.

Proposition 3.7

Any completely integrable distribution is involutive.

♠

Example 3.26

1. If $M = \mathbb{R}^n \times \mathbb{R}^k$ and $X_i = \frac{\partial}{\partial x_i}$ for $i = 1, \dots, n$, then $\mathcal{D} = \{X_1, \dots, X_n\}$ is an involutive distribution.

2. The Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G defines a left invariant involutive distribution.
3. All distributions on a two-dimensional manifold are involutive. However in higher dimension most distributions are not involutive. For example on $\mathbb{R}^3$ the distribution $\mathcal{D} = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\}$ is not involutive since $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right] = \frac{\partial}{\partial z}$.

Frobenius showed that being involutive is not only a necessary condition but also a sufficient one.

Theorem 3.5. (Frobenius)

A smooth distribution $\mathcal{D}$ on a manifold M is completely integrable if and only if it is involutive. ♡

Remark If the distribution has dimension 1, then this is nothing but the theorem on existence (and uniqueness) of solutions of a PDE. In this case an integral curve is an integral manifold. In dimension one however the necessary condition of being involutive is automatically satisfied since $[X, X] = 0$ for any vector field X.

Definition 3.14. Maximal integral submanifold

A maximal integral submanifold *N of an involutive distribution $\mathcal{D}$ on a manifold M is a connected integral manifold of $\mathcal{D}$ whose image in M is not a proper subset of any other connected integral manifold of $\mathcal{D}$. In other words, it is a connected integral manifold that contains any connected integral manifold with which it shares a point.* ♣

Theorem 3.6

Given an involutive distribution on a manifold M and a point $p \in M$, there exists a unique maximal integral manifold through p. ♡

For a proof of this and of Frobenius' Theorem, see for example [10].

Existence follows from the fact that in the Frobenius Theorem one can show that if the n-dimensional distribution is involutive and $p \in M$ is a point through which the integral manifold passes, then there exists a coordinate neighborhood (U, φ) with $U = (-\varepsilon, \varepsilon)^{n+k}$ centered at p, such that the integral manifold has shape $x_i =$ constant for $n \leq i + 1 \leq (n + k)$. One uses this coordinate neighborhood and the fact that M is second countable to patch the local "slices" of the integral manifolds to show the existence of a maximal one.

Recall from Lemma 3.1 that if a smooth map takes values in a regular submanifold N of a manifold M, then the same map thought as a map to N is also smooth. The same statement can be made for integral sumbanifolds of an involutive distribution.

Proposition 3.8

Let $\mathcal{D}$ be an involutive distribution on M and let N be a maximal integral submanifold. If $f\colon M' \to M$ is a smooth map and $f(M') \subset N$, then $f\colon M' \to N$ is also smooth. ♠

Proof Adapt the proof for regular submanifolds in Lemma 3.1 (Exercise 5.). □

We can finally prove Theorem 3.4.

Proof Let $X_1, \ldots, X_n$ be left invariant vector fields that form a basis for $\mathfrak{h}$. Since $\mathfrak{h}$ is a Lie algebra, the distribution $\mathcal{D} = \{X_1, \ldots, X_n\}$ is involutive and invariant under left translation by G. It follows that if N is an integral manifold of $\mathcal{D}$, then $L_g(N)$ is also an integral manifold of $\mathcal{D}$ for all $g \in G$. Let H be the unique maximal integral manifold through e. If $h \in H$, then $L_{h^{-1}}h = e$ hence both H and $L_{h^{-1}}H$ are maximal integral manifolds through e. By uniqueness $L_{h^{-1}}H = H$ and so $h^{-1}h' \in H$ for all $h' \in H$, that is H is a subgroup of G, as well as an immersed submanifold. Moreover since the maps $H \times H \to G$, $(h, h') \mapsto hh'$ and $H \to G$, $h \mapsto h^{-1}$ are smooth and take values in H, by the previous proposition the maps $H \times H \to H$ and $H \to H$ are smooth as well. Hence H is a Lie group, whose Lie algebra is $\mathfrak{h}$ by construction. The uniqueness follows from the uniqueness of the maximal integral manifold. □

One can generalize the above result.

Definition 3.15. Lie subgroup

Let G be a Lie group. We say that (H, φ) is a Lie subgroup *of G if*

1. *H is a Lie group;*
2. *$\varphi\colon H \to G$ is an injective Lie group homomorphism;*
3. *$\varphi(H)$ is an immersed submanifold, meaning φ is a one-to-one immersion.* ♣

Theorem 3.7. Lie group - Lie algebra correspondence II

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\tilde{\mathfrak{h}} \subset \mathfrak{g}$ a subalgebra. Then there exists a unique connected Lie subgroup (H, φ) of G such that $d\varphi(\mathfrak{h}) = \tilde{\mathfrak{h}}$, where $\mathfrak{h}$ is the Lie algebra of H. ♡

We have however seen that the topology of a Lie subgroup does not necessarily come from the topology of the ambient group. The following result tells us exactly when a Lie subgroup has the relative topology:

Theorem 3.8. Embedded Lie subgroups

Let (H, φ) be a Lie subgroup of a Lie group G. Then φ is an embedding if and only if $\varphi(H)$ is closed in G. ♡

We now move to the question of whether any Lie algebra homomorphism is the differential of a Lie group homomorphism.

Example 3.27 Let $\varphi\colon \mathbb{R} \to S^1$ be defined as $t \mapsto e^{it}$. Then $d_0\varphi\colon \mathrm{Lie}(\mathbb{R}) \to \mathrm{Lie}(S^1)$ is a Lie algebra isomorphism, and so is $(d_0\varphi)^{-1}\colon \mathrm{Lie}(S^1) \to \mathrm{Lie}(\mathbb{R})$. If $(d_0\varphi)^{-1}$ were the derivative of a homomorphism $\psi\colon S^1 \to \mathbb{R}$, then $\psi(S^1)$ would be a one-dimensional compact subgroup of $\mathbb{R}$. This is impossible since the only compact subgroup of $\mathbb{R}$ is the trivial one. Hence $(d_0\varphi)^{-1}$ does not come from a homomorphism. It does, however, come from a local homomorphism, namely the local inverse of φ.

We gave in Chapter 2 the definition of local homomorphism of topological groups. In the category of Lie groups the definition of local homomorphism has to be modified in that they are smooth maps (see also Theorem 3.14).

Theorem 3.9. From Lie algebra homomorphisms to local homomorphisms

1. *If G, H are Lie groups and $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism of the corresponding Lie algebras, then there exists a local homomorphism $\varphi\colon U \to H$ such that $\mathrm{d}_e\varphi = \pi$.*
2. *If π is a Lie algebra isomorphism then φ is a local isomorphism.* ♡

The proof of the second assertion follows immediately from the first one together with the following easy application of the Inverse Function Theorem, that will also be needed in the proof of the first assertion.

Lemma 3.5

If $\varphi\colon U \to H$ is a local homomorphism of Lie groups such that $d_e\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism, then φ is a local isomorphism. ♡

Proof Let U be a neighborhood of $e \in G$ such that φ is defined on U. Since $d_e\varphi$ is bijective, by the Inverse Function Theorem there exists a neighborhood U' of $e_G \in G$ and a neighborhood V of $e_H \in H$ such that $\varphi : U' \to V$ is a diffeomorphism. But then φ is a local isomorphism on

$U \cap U'$. □

Proof [Proof of Theorem 3.9] The important point is that, since π is a Lie algebra homomorphism, $\mathrm{Graph}(\pi)$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$. Indeed:

$$[(X, \pi(X)), (Y, \pi(Y))] = ([X, Y], [\pi(X), \pi(Y)]) = ([X, Y], \pi([X, Y])).$$

By Theorem 3.4 this implies that there exists a subgroup $K < G \times H$ such that $\mathrm{Lie}(K) = \mathrm{Graph}(\pi)$.

So far we have:

$$\mathrm{Graph}(\pi) \hookrightarrow \mathfrak{g} \times \mathfrak{h} \xrightarrow{d_e \mathrm{pr}_G} \mathfrak{g}$$

and

$$K \hookrightarrow G \times H \xrightarrow{\mathrm{pr}_G} G$$

Recall that we want to have a homomorphism *from* G, but here we have a homomorphism *to* G. By construction $\mathrm{pr}_G|_K : K \to G$ is a Lie group homomorphism and its derivative $d_e(\mathrm{pr}_G|_K) = \mathrm{pr}_{\mathfrak{g}}|_{\mathrm{Graph}(\pi)} \colon \mathrm{Graph}(\pi) \to \mathfrak{g}$ is a Lie algebra isomomorphism. Hence, by the previous lemma, $\mathrm{pr}_G|_K$ is a local isomorphism, that is, there exist neighborhoods $e_K \in W \subset K$ and $e_G \in V \subset G$ such that $\mathrm{pr}_G|_W \colon W \to V$ is an isomorphism. We consider then $(\mathrm{pr}_G|_W)^{-1} \colon V \to W$, whose derivative $d_e(\mathrm{pr}_G|_W)^{-1} \colon \mathfrak{g} \to \mathrm{Graph}(\pi)$ is $X \mapsto (X, \pi(X))$, by definition.

We consider now the homomorphism $\mathrm{pr}_H : G \times H \to H$ and its derivative $d_e \mathrm{pr}_H = \mathrm{pr}_{\mathfrak{h}} \colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, which is a Lie algebra homomorphism. Then $\mathrm{pr}_H \circ (\mathrm{pr}_G|_W)^{-1} \colon V \to H$ is the required local homomorphism, since

$$d_{e_G}(\mathrm{pr}_H \circ (\mathrm{pr}_G|_W)^{-1})(X) = d_{e_G}\mathrm{pr}_H \circ (d_{e_G}\mathrm{pr}_G|_W)^{-1}(X) = d_{e_G}\mathrm{pr}_H(X, \pi(X)) = \pi(X).$$

□

We remark again that the local homomorphism given by the theorem comes from the application of the Inverse Function Theorem in the previous lemma.

Theorem 3.10. (Ado)

Any finite-dimensional real Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n. ♡

The proof relies on the structure theory of Lie groups that we will see in the next chapter. We will give at that point a rough idea of the proof. Together with Theorem 3.9, this implies:

Corollary 3.4

Any Lie group G is locally isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$ *for some* n. ♡

Remark This result, seemingly very useful, is in practice not so. For example if we wanted to use the matrix bracket, we would need to know what is the n such that $\mathfrak{g} \hookrightarrow \mathfrak{gl}(n,\mathbb{R})$. Moreover the group G that we obtain might have a pretty ugly topology. However, together with the following corollary, which is an immediate consequence of Theorem 3.9, it will tell us for example under which conditions a Lie group can be isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$.

Corollary 3.5

1. *If G is a connected Lie group with Lie algebra $\mathfrak{g}$, there exists a simply connected Lie group $\widetilde{G}$ with Lie algebra isomorphic to $\mathfrak{g}$.*
2. *If two simply connected Lie groups have isomorphic Lie algebras, then they are isomorphic.*
3. *Given two isomorphic Lie algebras $\mathfrak{g}_1 \simeq \mathfrak{g}_2$, there exists simply connected isomorphic Lie groups $\widetilde{G}_i$ with $\mathrm{Lie}(\widetilde{G}_i) = \mathfrak{g}_i$, for $i = 1, 2$. In other words, there is a one-to-one correspondence between isomorphism classes of Lie algebras and isomorphism classes of simply connected Lie groups.* ♡

Proof (1) Using covering theory it is easy to show that if G is a connected Lie group, H a topological group and $p : H \to G$ a covering map, then there exists a unique Lie group structure on H such that p is a Lie group homomorphism and the kernel of p is a discrete subgroup of H. (See Exercise 6.)

(2) Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be simply connected Lie groups with $\mathfrak{g}_1 \simeq \mathfrak{g}_2$. By Theorem 3.9 there is a local isomorphism $p\colon U \to \widetilde{G}_2$. Since $\widetilde{G}_1$ is simply connected, by Theorem 2.1 p extends to a homomorphism $\widetilde{G}_1 \to \widetilde{G}_2$. Since this is also a covering map and $\widetilde{G}_2$ is simply connected, we have $\widetilde{G}_1 \simeq \widetilde{G}_2$.

(3) Let $\mathfrak{g}_1 \simeq \mathfrak{g}_2$. By Ado's Theorem there exists a Lie group G_i (locally isomorphic to a subgroup of $\mathrm{GL}(n_i,\mathbb{R})$) with $\mathrm{Lie}(G_i) = \mathfrak{g}_i$, for $i = 1, 2$. Let $\widetilde{G}_i$ be the universal covering. By (1) and (2) it follows that $\widetilde{G}_1 \simeq \widetilde{G}_2$. □

We conclude now with some easy consequences of the correspondence between Lie groups and Lie algebras.

Corollary 3.6. Abelian Lie groups and Lie algebras

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G is Abelian if and only if $\mathfrak{g}$ is Abelian. ♡

Proof ($\Rightarrow$) If G is Abelian, then $\mathrm{Inv}\colon G \to G : g \mapsto g^{-1}$ is a homomorphism (in fact, this condition is equivalent to being Abelian), so $d_e\mathrm{Inv}\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism by Proposition 3.6. We claim that $d_e\mathrm{Inv} = -Id$. Assuming this (which we will verify later), for all $X, Y \in \mathfrak{g}$:

$$-[X,Y] = d_e\mathrm{Inv}([X,Y]) = [d_e\mathrm{Inv}(X), d_e\mathrm{Inv}(Y)] = [-X,-Y] = [X,Y].$$

Therefore $[X,Y] = 0$ and so $\mathfrak{g}$ is Abelian.

To show the claim, let $\varphi\colon (-\varepsilon,\varepsilon) \to G$ be a path such that $\varphi(0) = e$ and let $\psi(t) := \mathrm{Inv}(\varphi(t))$. Then $\psi(0) = e$ and $e = \varphi(t)\psi(t)$, so that

$$0 = \left.\frac{d}{dt}\right|_{t=0} \varphi(t)\psi(t) = \varphi'(0)\psi(0) + \varphi(0)\psi'(0) = \varphi'(0) + \psi'(0).$$

If $\varphi'(0) = X \in \mathfrak{g}$, then $0 = X + d_e\mathrm{Inv}(X)$, which concludes the proof of the claim.

($\Leftarrow$) Suppose that $\mathfrak{g}$ is an Abelian Lie algebra, that is, it is isomorphic to the Lie algebra of $\mathbb{R}^n$ for some n. Then G is locally isomorphic to $\mathbb{R}^n$, that is, it is locally Abelian. But then G is Abelian by Proposition 2.1. 7. □

Corollary 3.7. Classification of connected Abelian Lie groups

1. *Any connected Abelian Lie group G is isomorphic to $\mathbb{T}^k \times \mathbb{R}^l$ for some $k, l \geq 0$.*
2. *Any compact connected Abelian Lie group is isomorphic to $\mathbb{T}^k$ for some $k \geq 0$.*
3. *Any simply connected Abelian Lie group is isomorphic to $\mathbb{R}^l$ for some $l \geq 0$.* ♡

Proof It will be enough to prove 1. and the other two statements will follow immediately. To this end, remark that the Lie algebra of G is isomorphic to the Lie algebra of $\mathbb{R}^n$ for some n. Hence by Theorem 3.9 there is a local isomorphism $\varphi\colon U_0 \to G$ for some open neighborhood $U_0 \subset \mathbb{R}^n$ of the origin. Since $\mathbb{R}^n$ is simply connected, Theorem 2.1 asserts that φ can be extended to a homomorphism $\varphi\colon \mathbb{R}^n \to G$.

We claim that $\ker\varphi$ is discrete. In fact, since $d_e\varphi$ is an isomorphism, there exists a neighborhood $e \in U_e \subset G$ such that $\varphi\colon U_0 \to \varphi(U_0) \cap U_e$ is a diffeomorphism. Then $\ker\varphi \cap U_0 = \{0\}$ and so $\{0\}$ is open in $\ker\varphi$. Since it is also closed, it is discrete. Thus there exist $x_1, \ldots, x_k \in \ker\varphi$ linearly independent over $\mathbb{R}$, such that $\ker\varphi$ is the $\mathbb{Z}$-span of $x_1, \ldots, x_k$. (See Exercise 9.) Let V be the $\mathbb{R}$-span of $x_1, \ldots, x_k$. Then $\dim V = k$ and we can write $\mathbb{R}^n = V \oplus W$, where $\dim W = (n-k)$. Now $\varphi\colon V \oplus W \to G$ is surjective, because $G = \bigcup_{n=1}^{\infty} (\varphi(U_0) \cap U_e)^n$ by Proposition 2.1 7. Therefore

$$G \cong V \oplus W/\ker\varphi = V/\ker\varphi \oplus W \cong (\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k} = \mathbb{T}^k \times \mathbb{R}^{n-k}.$$

□

3.5 The Exponential Map

p. 96 Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We introduce the exponential map of $\mathfrak{g}$ into G and study some of its properties. If $G = \mathrm{GL}(n,\mathbb{R})$ or one of its subgroups, then we will see that the exponential map coincides with the normal matrix exponential (from which the name follows). The exponential map is probably the most important basic construction associated to $\mathfrak{g}$ and G, as many important results in the general theory of Lie groups and Lie algebras depend in one way or another on the properties of this map.

Definition 3.16. One-parameter subgroup

Let G be a Lie group. A one-parameter subgroup *of G is a Lie group homomorphism $\varphi\colon \mathbb{R} \to G$ (i.e., a curve that is also a homomorphism).* ♣

Why would one-parameter subgroups exist? Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $X \in \mathfrak{g}$ and consider the Lie algebra homomorphism

$$\begin{aligned} \mathrm{Lie}(\mathbb{R}) &\to \mathfrak{g} \\ t &\mapsto tX \end{aligned}$$

By Theorem 3.9 there exists a local homomorphism that, since $\mathbb{R}$ is simply connected, according to Theorem 2.1 can be extended uniquely to a Lie group homomorphism $\varphi_X\colon \mathbb{R} \to G$ with the property that $d_0\varphi_X(t) = tX$.

Definition 3.17. Exponential map

The exponential map *of the Lie group G is defined by*

$$\begin{aligned} \exp_G\colon \mathfrak{g} &\to G \\ X &\mapsto \varphi_X(1)\,. \end{aligned}$$

♣

PICTURE

If $X \in \mathfrak{g}$ we denote by $\widetilde{X} \in \mathrm{Vect}(G)^G$ be the left invariant vector field with $\widetilde{X}_e = X$.

Proposition 3.9. Properties of the exponential map

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$.

1. *φ_X is an integral curve of $\widetilde{X}$ and the only one for which $\varphi_X(0) = e$.*
   More generally, $L_g\varphi_X \colon \mathbb{R} \to G$ is the only integral curve of $\widetilde{X}$ that goes through g at 0. In particular left invariant vector fields are always complete (that is, their integral curves are defined for all $t \in \mathbb{R}$).
2. *$\exp(tX) = \varphi_X(t)$ for all $t \in \mathbb{R}$ and hence $t \mapsto \exp(tX)$ is the unique one-parameter subgroup* corresponding to X, *that is $d_0(\exp(tX)) = d_0(\varphi_X(t)) = X$.*
3. *$\exp(t_1 + t_2)X = \exp(t_1X)\exp(t_2X)$ for all $t_1, t_2 \in \mathbb{R}$.*
4. *$\exp(tX)^{-1} = \exp(-tX)$ for all $t \in \mathbb{R}$.*
5. *$\exp : \mathfrak{g} \to G$ is a smooth map, and a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $e \in G$. In fact, $d_0 \exp = \mathrm{Id}$.*

Proof 1. The vector fields $\widetilde{1} \in \mathrm{Vect}(\mathbb{R})^{\mathbb{R}}$ and $\widetilde{X} \in \mathrm{Vect}(G)^G$ are φ_X-related (Lemma 3.3), so that φ_X is the unique integral curve of $\widetilde{X}$ such that $\varphi_X(0) = e$. Moreover $d_0\varphi_X(t) = \varphi'_X(t)$.

Since X is left invariant, $L_g \circ \varphi_X$ is also an integral curve and the unique one that goes through g at $t = 0$.

2. Let $t, s \in \mathbb{R}$, $X \in \mathfrak{g}$. We claim that $\varphi_{sX}(t) \overset{(*)}{=} \varphi_X(st)$. Assuming $(*)$ and setting $t = 1$, we obtain $\exp(sX) = \varphi_{sX}(1) = \varphi_X(s)$. Thus $s \mapsto \exp(sX) = \varphi_X(s)$ is the unique one-parameter subgroup whose tangent vector at $t = 0$ is X.

To prove $(*)$, recall that φ_{sX} is the unique integral curve of sX such that $\varphi_{sX}(0) = e$. On the other hand let $\eta\colon \mathbb{R} \to G$ be defined by $\eta(t) = \varphi_X(st)$ for $s \in \mathbb{R}$ fixed. Then $d_0\eta(t) = \frac{d}{dt}\big|_{t=0} \varphi_X(st) = s\varphi'_X(0) = sX$. Since $\eta(0) = e$, then η has the same properties as φ_{sX}, and $(*)$ follows.

3. and 4. are obvious since $t \mapsto \exp tX$ is a homomorphism.

5. Consider the manifold $M = G \times \mathfrak{g}$ and the *horizontal* vector field $\Xi \in \mathrm{Vect}(G \times \mathfrak{g})$ defined by

$$\Xi(g, X) = (\widetilde{X}_g, 0) \in T_gG \oplus T_X\mathfrak{g}\,.$$

Since $\widetilde{X}$ is smooth, Ξ is as well and hence there exists an integral curve $\varphi_{(\widetilde{X}_g,0)} \colon (-\epsilon, \epsilon) \to G \times \mathfrak{g}$ such that $L_g\varphi_{(\widetilde{X}_g,0)}(t) = (g \exp tX, X)$ is the integral curve through (g, X) at $t = 0$ (which is the translate under L_g of the integral curve through (e, X)). Since $G \times \mathfrak{g}$ is a Lie group, $\varphi_{(\widetilde{X}_g, 0)}$ is complete, so it is defined for all $t \in \mathbb{R}$, and in particular for $t = 1$, so that

$$\varphi_{(\widetilde{X}_g,0)}(1) = (g \exp X, X).$$

By the theorem on smooth dependence of solutions of ODEs on the initial conditions, $\varphi_{(\widetilde{X}_g,0)}$ is smooth on $G \times \mathfrak{g}$. Let now $\mathrm{pr}_G : G \times \mathfrak{g} \to G$ be the projection, which is a smooth map. Then $\exp : \mathfrak{g} \to G$ can be written as

$$\exp(X) = \mathrm{pr}_G \circ \varphi_{(X,0)}(1).$$

and is hence smooth.

To check that exp is a local diffeomorphism it is enough to check that $d_0 \exp \colon T_0\mathfrak{g} \to T_eG$ is invertible. We show in particular that $d_0 \exp = \mathrm{Id}$. Let $\psi \colon (-\epsilon, \epsilon) \to \mathfrak{g}$ be the curve $\psi(t) = tX$. Then $\psi(0) = 0$ and $\psi'(0) = X$. But $\varphi_X \colon \mathbb{R} \to G$, $\varphi_X(t) = \exp(tX)$ has the property that $\varphi_X(0) = 0$ and $\varphi'_X(0) = X$, so that $d_0\varphi_X = \psi$. Thus $d_0\varphi_X(t) = tX$, so that $d_0 \exp(tX) = tX$. □

For a Lie group this amounts to saying that there is one curve that behaves well, namely it is a homomorphism $\mathbb{R} \to G$. What the exponential map does is to take a line and push it down to the group wrapping it around and preserving the group structure.

Example 3.28

1. Let $G = \mathbb{R}^n$. Then $\exp : \mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n \to \mathbb{R}^n$ is the identity.
2. PICTURE Let $G = S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then $T_1S^1 \cong i\mathbb{R} \cong \mathbb{R}$ and, chasing the definitions, it is easy to see that $\exp \colon \mathbb{R} \to S^1$ is $\exp(t) = e^{it}$.

We want to identify the exponential map for $G = \mathrm{GL}(n, \mathbb{C})$ and we will see that it is indeed equal to the usual matrix exponential. We will need the following:

Lemma 3.6. Matrix exponential

Let V be an n-dimensional complex vector space.

1. *The map $X \mapsto e^X := \sum_{j=0}^{\infty} \frac{X^j}{j!}$ is a well defined map from $\mathrm{End}(V)$ to $\mathrm{GL}(V)$.*
2. *$\det(e^X) = e^{\mathrm{tr} X}$ for all $X \in \mathrm{End}(V)$.*
3. *If X and Y commute, then $e^{X+Y} = e^X e^Y$.*

♡

Proof 1. To see that the map is well defined we will show that the right hand side converges uniformly on compact sets. In fact let $K \subset \mathrm{End}(V)$ be a compact set and let $c > 0$ be such that

$|X_{ij}| \leq c$ if $X \in K$. Then an induction argument shows that $|(X^m)_{ij}| \leq (nc)^m$. Since $\sum\limits_{m=0}^{\infty} \frac{(nc)^m}{m!}$ converges, $\sum\limits_{m=0}^{\infty} \frac{(X^m)_{ij}}{m!}$ converges uniformly on K, and hence the same is true of $\sum\limits_{m=0}^{\infty} \frac{X^m}{m!}$. (Here we used the Weierstrass test: if (f_n) is a sequence of real or complex valued functions on a set A with the property that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum\limits_{n=0}^{\infty} M_n$ converges, then the series $\sum\limits_{n=0}^{\infty} f_n(x)$ converges uniformly on A.)

Hence the map is well defined and we have to see that it takes values in $\mathrm{GL}(V)$, that is, it is invertible. In fact, let $S_j(X)$ be the j-th partial sum $\sum\limits_{m=0}^{j} \frac{X^m}{m!}$. By continuity of the multiplication $\mathrm{End}(V) \to \mathrm{End}(V)$, $X \mapsto BX$ for $B \in \mathrm{End}(V)$ we have that $B \lim\limits_{j\to\infty} S_j(X)B^{-1} = \lim\limits_{j\to\infty} BS_j(X)B^{-1}$, so that

$$Be^X B^{-1} \overset{(*)}{=} e^{BXB^{-1}}. \tag{3.6}$$

Remark now that we can find $B \in \mathrm{GL}(V)$ such that BXB^{-1} is upper triangular: in fact, if v_1 is an eigenvector of X, we can construct inductively v_{j+1} as an eigenvector of $\mathrm{pr}_j \circ X$, where $\mathrm{pr}_j : V \to W_j$ is the projection onto W_j where $V = V_j \oplus W_j$ and $V_j = \mathrm{span}\{v_1, \dots, v_j\}$. Now choose B to be the matrix that has v_j as column vectors, and let λ_j be the eigenvalue corresponding to v_j.

If $BXB^{-1} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, then

$$e^{BXB^{-1}} = \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix},$$

so that

$$\det e^X = \det Be^X B^{-1} \overset{(*)}{=} \det e^{BXB^{-1}} \neq 0,$$

and so $e^X \in \mathrm{GL}(V)$.

2. In particular we have that if BXB^{-1} is upper-triangular, then

$$\det e^{BXB^{-1}} = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \dots + \lambda_n} = e^{\mathrm{tr}(BXB^{-1})}.$$

By invariance under conjugation, and using again (3.6),

$$\det e^X = \det Be^X B^{-1} = \det e^{BXB^{-1}} = e^{\mathrm{tr} BXB^{-1}} = e^{\mathrm{tr} X}.$$

3. We can write:

$$e^X e^Y = \left(\sum_{j=0}^{\infty} \frac{X^j}{j!}\right)\left(\sum_{\ell=0}^{\infty} \frac{Y^\ell}{\ell!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{X^k Y^{n-k}}{k!(n-k)!} =$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k} = \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!} = e^{X+Y}\,.$$

□

Corollary 3.8. Exponential map of $\mathrm{GL}(V)$

The assignment $t \mapsto e^{tX}$ is a smooth curve in $\mathrm{GL}(V)$ that takes the value I at $t=0$ and such that the tangent vector at I is X. Hence $e^{tX} = \exp(tX)$, so that $\exp\colon \mathfrak{gl}(V) \to \mathrm{GL}(V)$ is just $\exp(X) = e^X$. ♡

Proof From the previous lemma we deduce that $t \mapsto e^{tX}$ is a homomorphism. Since

$$\frac{d}{dt} e^{tX}\Big|_{t=0} = \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\Big|_{t=0} = X \sum_{m=1}^{\infty} \frac{m}{m!}(tX)^{m-1}\Big|_{t=0}$$
$$= X \sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\Big|_{t=0} = X e^{tX}\big|_{t=0} = X,$$

by uniqueness we have the assertion. □

Proposition 3.10. Naturality of $\exp$

Let $\varphi\colon G \to H$ be a Lie group homomorphism. Then the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "d_e\varphi"] \arrow[d, "\exp_G"'] & \mathfrak{h} \arrow[d, "\exp_H"] \\
G \arrow[r, "\varphi"] & H\,,
\end{tikzcd}$$

that is $\varphi \circ \exp_G = \exp_H \circ d_{e_G}\varphi$. ♠

Proof Let $X \in \mathfrak{g}$. Then $t \mapsto \exp_G(tX)$ is the unique one-parameter subgroup of G that takes the value e_G and has tangent vector X at $t=0$. Since φ is a homomorphism, then $t \mapsto \varphi(\exp_G(tX))$ is a one-parameter subgroup of H that takes the value e_H at $t=0$ and whose tangent vector at $t=0$ is

$$\frac{d}{dt}\Big|_{t=0} \varphi(\exp_G(tX)) = d_{e_G}\varphi(X)\,.$$

But the only one-parameter subgroup of H with these properties is $t \mapsto \exp_H(t\, d_{e_G}\varphi(X))$. Hence for all $t \in \mathbb{R}$ we have $\varphi(\exp(tX)) = \exp(t\, d_{e_G}\varphi(X))$, so that $\varphi \circ \exp = \exp \circ d_{e_G}\varphi$. □

Since $\mathfrak{g}$ is connected, then $\exp(\mathfrak{g}) \subseteq G^0$, the connected component of G, but there is not necessarily equality. In other words, $\exp\colon \mathfrak{g} \to G$ may not be surjective, even when G is connected.

Example 3.29 We want to show that the exponential map is not necessarily surjective, even for connected Lie groups. Let $G = \mathrm{SL}(2,\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. We show that $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is not surjective.

The argument is in two steps:

1. We first show that the image $\exp(\mathfrak{sl}(2,\mathbb{R}))$ consists of matrices that are squares. That is if $A \in \exp(\mathfrak{sl}(2,\mathbb{R}))$, then $A = B^2$ for some $B \in \exp(\mathfrak{sl}(2,\mathbb{R}))$.
2. We will show that there exists $A \in \mathrm{SL}(2,\mathbb{R})$ that is not a square. We will do this by showing that if $A \in \mathrm{SL}(2,\mathbb{R})$ is a square, then $\mathrm{tr}(A) \geq -2$. It follows that $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ misses the whole open set $\{A \in \mathrm{SL}(2,\mathbb{R}) : \mathrm{tr}(A) < -2\}$.

1.Let $X \in \mathfrak{sl}(2,\mathbb{R})$ and let $A := \exp(X)$. Then

$$A = \exp(X) = \exp\left(\frac{X}{2} + \frac{X}{2}\right) = \exp\left(\frac{X}{2}\right)^2 = B^2,$$

where $B = \exp\left(\frac{X}{2}\right)$. Notice that if $X \in \mathfrak{sl}(2,\mathbb{R})$, then $\mathrm{tr}(X) = 0$, so $\mathrm{tr}\left(\frac{X}{2}\right) = 0$, which implies that $B \in \exp(\mathfrak{sl}(2,\mathbb{R}))$ as well.

2. Any matrix $B \in \mathrm{GL}(2,\mathbb{R})$ is a root of its characteristic polynomial $\lambda^2 - \mathrm{tr}(B)\lambda + \det(B)$, that is

$$B^2 - \mathrm{tr}(B)B + \det(B)I = 0.$$

By taking the trace of this equation, and assuming further that $B \in \mathrm{SL}(2,\mathbb{R})$, we obtain

$$\mathrm{tr}(B^2) - \mathrm{tr}(B)^2 + 2 = 0.$$

Letting $A = B^2$, this implies that

$$\mathrm{tr}(A) = \mathrm{tr}(B)^2 - 2 \geq -2\,.$$

But $\{A \in \mathrm{SL}(2,\mathbb{R}) : \mathrm{tr}(A) < -2\}$ is not empty since for example if $A = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, then $\mathrm{tr}(A) < -2$. Thus $\exp_{\mathrm{SL}(2,\mathbb{R})}$ is not surjective.

In the previous example the non-compactness of $\mathrm{SL}(2,\mathbb{R})$ plays an important role, as the following theorem shows:

Theorem 3.11. Cartan

The exponential map of a compact connected Lie groups is surjective. ♡

Remark For a non-compact connected Lie group G, the next best thing to surjectivity is the following: every $g \in G$ can be written as $\exp X_1 \cdots \exp X_n$ for $X_1, \ldots, X_n \in \mathfrak{g}$ (von Neumann, 1929). The proof is as in the extension of a local homomorphism (Theorem 2.1). A more recent result shows that in fact $n = 2$ is sufficient.

The proof of Cartan's Theorem is not difficult, but it relies on results either in differential geometry or in the structure theory of Lie groups that we have not yet covered. We give here the idea of two proofs. For a complete proof see for example [2, Chapter 16, 17].

1. Any compact connected Lie group can be given the structure of a Riemannian manifold with a bi-invariant metric. (In fact, this is almost a characterization, in the sense that a Lie group admits a bi-invariant metric if and only it is the product of a compact Lie group and an Abelian Lie group.) This can be done in either one of two ways: either by averaging an arbitrary positive definite inner product on $\mathfrak{g}$ and then translating it to a left invariant positive definite inner product on T_gG, that is to a bi-invariant metric on G; or by embedding G into $\mathrm{U}(n)$ using the Peter–Weyl Theorem, and obtaining a bi-invariant metric on G from the one on $\mathbb{C}^{n\times n}$.

In either cases, since G is compact and connected, hence complete, one can use the Hopf-Rinow Theorem and deduce that any two points are joined by a geodesic. It follows that the Riemannian exponential, obtained by following geodesics, is surjective. One can then prove that, under the given settings and structure on G, the Riemannian exponential coincides with the Lie group exponential.

2. One can easily see that the exponential map of a torus is surjective. Then one can show that in a compact connected Lie group, every element lies in a maximal torus, and all tori are conjugate.

Example 3.30 Let

$$N_1 = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{SL}(n, \mathbb{R}) \right\},$$

with Lie algebra

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{R}) \right\}.$$

If $A \in \mathfrak{n}_1$, then $A^n = 0$, so $\exp(A) = \sum_{j=0}^{n-1} \frac{A^j}{j!}$. Moreover, if $B \in N_1$, then we can write $B = I + B'$ where $(B')^n = 0$. Define then $\log\colon N_1 \to \mathfrak{n}_1$, by

$$\log(B) = \log(I + B') := \sum_{j=1}^{n-1} (-1)^{j-1} \frac{(B')^j}{j}.$$

Finite power series manipulation shows that $\exp\colon \mathfrak{n}_1 \to N_1$ and $\log\colon N_1 \to \mathfrak{n}_1$ are inverse of each other, and so $\exp$ is surjective.

We now see some applications of the exponential map. We saw that if $X, Y \in \mathrm{GL}(n, \mathbb{C})$ commute and e^X is the matrix exponential then $e^{X+Y} = e^X e^Y$.

Proposition 3.11

If G is a connected Abelian group, then $\exp\colon \mathfrak{g} \to G$ is a group homomorphism, that is

$$\exp(X + Y) = \exp(X)\exp(Y)$$

for every $X, Y \in \mathfrak{g}$ and $G \cong \mathfrak{g}/\Gamma$, where $\Gamma := \ker \exp$ is discrete. ♠

Notice that we could recover here immediately the classification of connected Abelian Lie groups (Corollary 3.7).

Proof Since G is Abelian, the multiplication map $m\colon G \times G \to G$ is a homomorphism. In fact

$$m(g_1, h_1)m(g_2, h_2) = g_1 h_1 g_2 h_2 = g_1 g_2 h_1 g_2 = m(g_1 g_2, h_1 h_2)$$

and we saw already that $d_{(e,e)}m(X, Y) = X + Y$. Using these two facts and the naturality of the exponential map we have that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "d_{(e,e)}m"] \arrow[d, "\exp_{G\times G}"'] & \mathfrak{g} \arrow[d, "\exp_G"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

In other words

$$\begin{aligned}\exp_G(X+Y) &= \exp_G d_{(e,e)}m(X,Y) = m \exp_{G\times G}(X,Y) \\ &= m(\exp_G(X), \exp_G(Y)) = \exp_G(X)\exp_G(Y)\,,\end{aligned}$$

where we used that, if G and H are Lie groups, $\exp_{G\times H} = \exp_G \times \exp_H$.

Since $\exp$ is a homomorphism, its image is a subgroup containing a neighborhood of the identity. By Proposition 2.1.7., the exponential map is surjective. The fact that $\ker\exp$ is discrete follows from the fact that it is a local diffeomorphism. □

We can now prove a characterization of Lie groups related to Hilbert's fifth problem. In 1900 Hilbert formulated 23 problems, the fifth of which, reinterpreted in modern terminology, contains the question as to whether it makes a difference to require in the definition of a Lie group that it is a topological manifold or a smooth manifold. This was proven in the negative (as expected) at the beginning of the 50s by Gleason [3], Montgomery–Zippin [5] and Yamabe [11]. The possible existence of small subgroups was recognized as one of the main difficulties involved in the proof.

Definition 3.18. Small subgroup

A topological group G is said to have small subgroups *if every neighborhood of the identity contains a non-trivial subgroup.* ♣

Theorem 3.12. No small subgroups

A connected locally compact topological group is a Lie group if and only if it has no small subgroups. ♡

We are not going to prove the "if" direction of the theorem in general. We are only going to illustrate it with the following two results. A good reference in modern terminology is [9, Theorem 1.1.13].

Lemma 3.7

Let $L := \prod_{n\geq 1} G_n$ be an infinite product of non-trivial topological groups. Then L has small subgroups. ♡

Proof Let V be a neighborhood of e in L. By definition of the product topology there exists $k \geq 1$ and neighborhoods $V_1, \dots, V_k$ of e in $G_1, \dots, G_k$ such that $V \supset \prod_{i=1}^{k} V_i \times \prod_{j=k+1}^{\infty} G_j$. Hence V contains the group $(e, \dots, e) \times \prod_{j=k+1}^{\infty} G_j$. □

For compact groups the fact that the condition of having no small subgroups is sufficient was already proven by Von Neumann:

Theorem 3.13. (Von Neumann)

If a compact topological group K has no small subgroups then it is a Lie group. ♡

Proof According to Peter–Weyl Theorem, the left regular representation of K decomposes as a (not necessarily countable) direct sum of finite dimensional irreducible unitary representations. Let us consider the continuous injective homomorphism

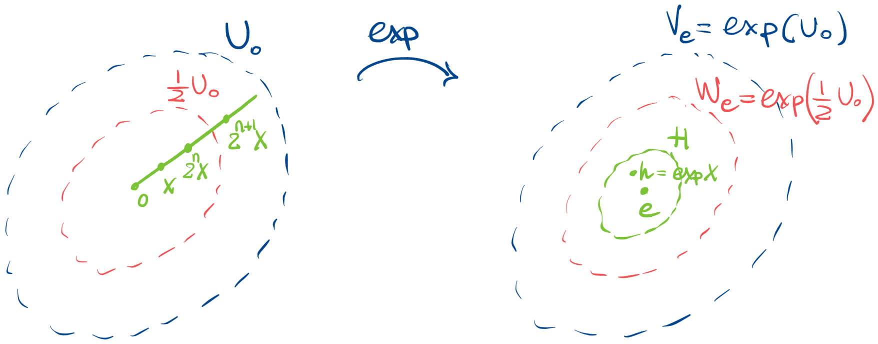
$$\begin{aligned} \Lambda\colon K &\to \prod_{\rho\in\hat{K}} U(\Lambda_\rho) \\ x &\mapsto \quad (\rho(x))_{\rho\in\hat{K}}\,, \end{aligned}$$

where $\hat{K}$ is the unitary dual of K and the $U(\Lambda_\rho)$ are compact groups. If $\Lambda(K)$ has no small subgroups, then there exists $n \geq 1$ such that

$$\Lambda(K) \cap \left\{ (e_{\rho_1}, \ldots, e_{\rho_n}) \times \prod_{\substack{\rho\in\hat{K} \\ \rho\neq\rho_1,\ldots,\rho_n}} U(\Lambda_\rho) \right\} = \{e\}\,.$$

But then $\mathrm{pr}\colon \Lambda(K) \to U(\Lambda_{\rho_1}) \times \cdots \times U(\Lambda_{\rho_n})$ is injective. Therefore K can be embedded as a closed subgroup of a finite dimensional unitary group, hence it is a Lie group. □

Proof [Proof of Theorem 3.12 ($\Rightarrow$)] Let $0 \in U_0 \subset \mathfrak{g}$ and $e \in V_e \subset G$ be open neighborhoods such that $\exp\colon U_0 \to V_e$ is a diffeomorphism, and let $W_e := \exp \frac{1}{2} U_0$. We will show that W_e does not contain non-trivial subgroups $H < G$.



Suppose by contradiction that $\{e\} \neq H$ is a subgroup of G such that $H \subset W_e$. Let $e \neq h \in H$ and $X \in \frac{1}{2} U_0$ such that $\exp X = h$. We will show that there are powers of h not in H, contradicting that H is a subgroup. In fact, let $n \in \mathbb{N}$ be such that $2^n X \in \frac{1}{2}U_0$ and $2^{n+1}X \notin \frac{1}{2}U_0$. Notice that, since $2^n X \in \frac{1}{2}U_0$, then $2^{n+1}X \in U_0$. Then by Proposition 3.11

$$h^{2^{n+1}} = \exp(2^{n+1}X) \in \exp\left(U_0 \setminus \frac{1}{2}U_0\right) \subseteq V_e \setminus W_e\,.$$

So $h^{2^{n+1}} \notin H \subset W_e$, which is a contradiction. □

Hence, given a topological group G, there is a criterion to determine whether or not the group G can be made into a Lie group. A natural question is then whether a Lie group can have several smooth structures. The answer is a corollary of the following theorem:

Theorem 3.14. Continuous implies smooth

Any continuous homomorphism of two Lie groups is smooth. ♡

Corollary 3.9

Two real Lie groups that are isomorphic as topological groups are isomorphic as Lie groups. ♡

Notice that the assumption the the groups are real is essential. In fact $\mathbb{C}/(\mathbb{Z}+\imath\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z}+2\imath\mathbb{Z})$ are isomorphic as real Lie groups but not as complex Lie groups.

Proof [Proof of Corollary 3.9] The continuous isomorphism would be a diffeomorphism between the two smooth structures. □

We start the proof of Theorem 3.14 with the following.

Proposition 3.12. (Local coordinates)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ be subspaces such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Then the map

$$\begin{aligned} \phi\colon \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k &\longrightarrow G \\ X_1 + \cdots + X_k &\mapsto \exp_G(X_1)\cdots\exp_G(X_k) \end{aligned}$$

is a smooth map and a local diffeomorphism $\phi\colon W_1 + \cdots + W_k \to V_e$, where $0 \in W_i \subset \mathfrak{g}_i$ and $e \in V_e \subset G$ are open neighborhoods. In fact $d_0\phi = \mathrm{Id}$. ♠

Proof We take for simplicity $k = 2$, we identify $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 \times \mathfrak{g}_2$ and we set $\exp_i := \exp_G|_{\mathfrak{g}_i}$

so that $d_0 \exp_i = \mathrm{Id}_{\mathfrak{g}_i}$. Then

$$\begin{aligned} d_0\phi(X_1+X_2) = d_{(0,0)}\phi(X_1,X_2) &= d_0 m(\exp_1 \times \exp_2)(X_1,X_2) \\ &= d_{(e,e)}m(d_0 \exp_1 X_1, d_0 \exp_2 X_2) \\ &\overset{(*)}{=} d_{(e,e)}m(X_1,X_2) = X_1+X_2\,, \end{aligned}$$

where in $(*)$ we used Proposition 3.9.5. □

Remark As a consequence of the proposition, if $\{X_1,\dots,X_n\}$ is a basis of $\mathfrak{g}$, and

$$U := \left\{ (t_1,\dots,t_n) \in \mathbb{R}^n : \sum_{j=1}^n t_j X_j \in W_1 + \cdots W_n \right\} ,$$

the map

$$\begin{aligned} V_e &\longrightarrow U \\ \phi\left(\sum_{j=1}^n t_j X_j\right) = \prod_{j=1}^n \exp(t_j X_j) &\mapsto (t_1,\dots,t_n) \end{aligned}$$

gives a chart at the identity in G. Using this coordinate chart and left translation, we can construct an atlas for G.

Proof [Proof of Theorem 3.14] Let $h\colon G \to H$ be a continuous homomorphism of Lie groups, $\{X_1,\dots,X_n\}$ is a basis of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, where $\mathfrak{g}_i = \mathbb{R}X_i$. Let $\exp_i := \exp_G|_{\mathfrak{g}_i}$ and

$$\begin{aligned} \phi\colon \quad \mathfrak{g} &\longrightarrow G \\ X_1+\cdots+X_n &\mapsto (\exp_1 X_1)\cdots(\exp_n X_n)\,. \end{aligned}$$

or

$$\begin{aligned} \phi\circ\psi\colon \quad U &\longrightarrow G \\ (t_1,\dots,t_n) &\mapsto \exp_1(t_1X_1)\cdots\exp_n(t_nX_n) \end{aligned}$$

where

$$\begin{aligned} \psi\colon \quad U &\longrightarrow W_1+\cdots+W_n \subset \mathfrak{g} \\ (t_1,\dots,t_n) &\mapsto t_1X_1+\cdots t_nX_n\,. \end{aligned}$$

We assume now the following:

Proposition 3.13

Any continuous homomorphism $\mathbb{R} \to G$ is smooth. ♠

Collecting the above maps, we have that

$$\begin{array}{ccccccc} U & \xrightarrow{\psi} & W_1 + \cdots + W_n & \xrightarrow{\phi} & V_e & \xrightarrow{h} & H \\ (t_1,\dots,t_n) & \mapsto & t_1X_1 + \cdots + t_nX_n & \mapsto & \prod_{i=1}^n \exp(t_iX_i) & \mapsto & h\left(\prod_{i=1}^n \exp(t_iX_i)\right). \end{array}$$

Since h is a homomorphism

$$h\left(\prod_{i=1}^n (\exp(t_iX_i)\right) = \prod_{i=1}^n h(\exp(t_iX_i)).$$

Let now $h_i\colon \mathbb{R} \to G$ be defined as $h_i(t) := h(\exp(tX_i))$. Then h_i is a continuous homomorphism, hence smooth by Proposition 3.13. Thus $\prod_{i=1}^n h(\exp(t_iX_i)) = \prod_{i=1}^n h_i(t_i)$ and since $\phi \circ \psi$ is a diffeomorphism, h is smooth at the origin and hence smooth everywhere. □

Hence we are left to prove the proposition, which is where the heart of the issue lies.

Proof [Proof of Proposition 3.13] Let $0 \in W_0 \subset \mathfrak{g}$ and $e \in U_e \subset G$ be neighborhoods such that $\exp_G : W_0 \to U_e$ is a diffeomorphism, and let $e \in V_e \subset G$ be such that $V_e^2 \subset U_e$. Note that $V_e \subset V_e^2$.

Claim 3.5.1. *Every element $g \in V_e$ has a unique square root*

$$\sqrt{g} = \exp\left(\frac{1}{2}\exp^{-1}(g)\right).$$

In other words, the map $V_e \to U_e$ defined by $g \mapsto g^2$ is injective and the image contains V_e.

In fact, if $g \in V_e \subset U_e$, let $X \in W_0$ be such that $\exp(X) = g$, and let us consider the one-parameter subgroup of G defined by $\varphi_X(t) = \exp(tX)$. Then $\varphi_X(1) = \exp(X) = g$ and

$$g^2 = \varphi_X(1)^2 = \varphi_X(2) = \exp(2X) = \exp(2\exp^{-1}(g)).$$

But also $g^2 \in U_e$, where $\exp^{-1}$ is a diffeomorphism, so

$$\exp^{-1}(g^2) = 2\exp^{-1}(g) \Rightarrow \frac{1}{2}\exp^{-1}(g^2) = \exp^{-1}(g) \Rightarrow \exp\left(\frac{1}{2}\exp^{-1}(g^2)\right) = g$$

which proves the claim.

To conclude the proof it will be enough to prove the following:

Claim 3.5.2. *Let $h\colon \mathbb{R} \to G$ be a continuous homomorphism and let $X \in \mathfrak{g}$ be such that $\exp(X) = h(1)$, then $h\left(\frac{1}{2^n}\right) = \exp\left(\frac{1}{2^n}X\right)$.*

In fact, assuming the claim, if $p \in \mathbb{Z}$ then

$$h\left(\frac{p}{2^n}\right) = h\left(\frac{1}{2^n}\right)^p = \exp\left(\frac{1}{2^n}X\right)^p = \exp\left(\frac{p}{2^n}X\right).$$

So the assertion holds for all dyadic numbers. Since these are dense in $\mathbb{R}$, the conclusion of Proposition 3.13 follows from the continuity of h.

We show the assertion in Claim 3.5.2 by induction. We start by justifying why the hypothesis holds. In fact, since h is continuous and V_e is open, there exists $\varepsilon > 0$ such that if $|t| \le \varepsilon$, then $h(t) \in V_e$. By rescaling h (that is considering $h_\epsilon(t) := h(t/\epsilon)$), we may assume that $\varepsilon = 1$, so that $h(1) \in V_e$.

For the base case $n = 1$, set $g_0 := h(1) \in V_e$ and $X := \exp^{-1}(g_0)$. Then by Claim 3.5.1:

$$\sqrt{g_0} = \exp\left(\frac{1}{2}\exp^{-1}(g_0)\right) = \exp\left(\frac{1}{2}X\right)$$

is the unique square root of $g_0 \in V_e$. But also $g_0 = h(1) = h\left(\frac{1}{2}\right)^2$, so by uniqueness of the square root in Claim 3.5.1, $h\left(\frac{1}{2}\right) = \sqrt{g_0} = \exp\left(\frac{1}{2}X\right)$.

Now assume that for all $k < n$ we have $h\left(\frac{1}{2^k}\right) = \exp\left(\frac{1}{2^k}X\right)$. Set $g_n := h\left(\frac{1}{2^n}\right) \in V$. By Claim 3.5.1 again:

$$\begin{aligned} h\left(\frac{1}{2^n}\right) = g_n &= \exp\left(\frac{1}{2}\exp^{-1}(g_n^2)\right) \\ &= \exp\left(\frac{1}{2}\exp^{-1}\left(h\left(\frac{1}{2^n}\right)^2\right)\right) = \exp\left(\frac{1}{2}\exp^{-1}\left(h\left(\frac{1}{2^{n-1}}\right)\right)\right) \\ &= \exp\left(\frac{1}{2}\exp^{-1}\exp\left(\frac{1}{2^{n-1}}X\right)\right) = \exp\left(\frac{1}{2^n}X\right). \end{aligned}$$

This concludes the proof. □

The following theorem was already mentioned but we can finally prove it. It was first shown by Von Neumann for $G = \mathrm{GL}(n, \mathbb{R})$, then extended to all Lie groups by Cartan.

Theorem 3.15. Closed Subgroup Theorem

Let G be a real Lie group and H a closed subgroup. Then H is a real Lie group. ♡

Remark The result does not hold if G is a complex Lie group, that is, if it is locally diffeomorphic to $\mathbb{C}^n$ for some n. The point is that being a submanifold does not force a complex structure on the group.

Example 3.31 $\mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^\times$ is a complex Lie group, but S^1 is a closed subgroup that is not a complex Lie group, for instance because it is odd-dimensional. It is however a real Lie group.

The proof will rely upon the following two lemmas that are left as an exercise.

Lemma 3.8

Let H be an abstract subgroup of the Lie group G and let $\mathfrak{h}$ be a subspace of $\mathfrak{g} = Lie(G)$. Let $0 \in U_0 \subset \mathfrak{g}$ and $e \in V_e \subset G$ be open neighborhoods such that $\exp : U_0 \to V_e$ is a diffeomorphism. Suppose that

$$\exp(U_0 \cap \mathfrak{h}) = V_e \cap H\,. \tag{3.7}$$

Then:

1. *H is a Lie subgroup of G with the induced topology;*
2. *$\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;*
3. *$\mathfrak{h} = \mathrm{Lie}(H)$.*

♡

Idea of the proof: Use (3.7) to define charts on H and use that if $X \in \mathfrak{g}$, then $X \in \mathfrak{h}$ if and only if $\exp tX \in H$ for all $t \in \mathbb{R}$ (see Warner, p.104).

We saw that given a Lie group G, the pair (H, φ) is a Lie subgroup if

1. H is a group;
2. H is a manifold, and the the smooth structure is compatible with the group structure;
3. $\varphi(H)$ is a submanifold of G.

In fact, the requirement of the compatibility of the group structure and the manifold structure of H is not necessary, as one can prove the following:

Fact: If H is a subgroup of a Lie group G admitting a manifold structure that makes it into a submanifold of G, then the manifold and the group structure are compatible. Hence H (or $\varphi(H)$) is a Lie subgroup of G.

Sketch of proof. Take T_eH and consider the distribution $\mathcal{D}$ obtained by left translation via elements of G. Then one can show that H is an integral manifold of $\mathcal{D}$ and hence $\mathcal{D}$ is an involutive distribution. One can then show that the group operations are smooth (Warner, p. 195).

In fact, (3.8) is exactly what is needed to give H a manifold structure. The fact that the topology of H coincides with the topology induced by G follows from the fact that H is closed in G.

Lemma 3.9

Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then for t small enough

$$\exp(tX)\exp(tY) = \exp\big(t(X+Y) + O(t^2)\big)\,, \tag{3.8}$$

where $\frac{1}{t^2}O(t^2)$ is bounded at $t = 0$.

♡

Proof [Proof of Theorem 3.15] We want to identify a subspace $\mathfrak{h}$ of $\mathfrak{g}$ that will satisfy (3.7). To this purpose, let

$$\mathfrak{h} := \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}\,.$$

Then

1. we first show that (3.8) implies that $\mathfrak{h}$ is a subspace,
2. then that $\mathfrak{h}$ and H satisfy (3.7) for appropriate U_0 and V_e.

To see that $\mathfrak{h}$ is a subspace, observe that it is closed under multiplications by scalars. To see that $\mathfrak{h}$ is closed under addition, observe that it follows from (3.8) it follows

$$(\exp(tX)\exp(tY))^n = \left(\exp\left(t(X+Y)+O(t^2)\right)\right)^n = \exp\left(nt(X+Y)+O(nt^2)\right).$$

Replacing t by $\frac{t}{n}$, one obtains

$$\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n = \exp\left(t(X+Y)+\frac{1}{n}O(t^2)\right),$$

which shows that

$$\lim_{n\to\infty}\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n = \exp(t(X+Y)).$$

Thus, since H is a closed subgroup, if $X, Y \in \mathfrak{h}$ then also $X+Y \in \mathfrak{h}$. Hence $\mathfrak{h}$ is a subspace.

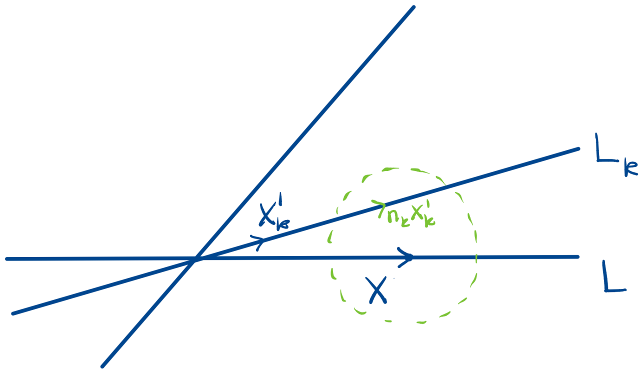
To show that H and $\mathfrak{h}$ satisfy (3.7) we proceed by contradiction. Since $\exp(U_0 \cap \mathfrak{h}) \subseteq V_e \cap H$, if (3.7) did not hold, for every $U_0 \subset \mathfrak{g}$ and $V_e \subset G$ such that $\exp\colon U_0 \to V_e$ is a diffeomorphism, we could find $h \in V_e \cap H$ but $h \notin \exp(U_0 \cap \mathfrak{h})$. Thus let $0 \in W_0 \subset \mathfrak{h}$ be an open neighborhood and $(h_k)_{k\geq 1} \subset H$ a sequence such that $h_k \to e$ and $h_k \notin \exp(W_0)$.

Let now $\mathfrak{h}'$ be a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ and, using Proposition 3.12, choose $0 \in N_0 \subset W_0 \subset \mathfrak{h}$ and $0 \in N_0' \subset \mathfrak{h}'$ such that the map

$$\begin{aligned} N_0 \times N_0' &\longrightarrow \qquad G \\ (X, X') &\mapsto \exp X \exp X' \end{aligned}$$

is a diffeomorphism onto its image $A_e \ni e$.

By assumption $h_k \to e$ and so $h_k \in A_e$ for all large k, so that $h_k = \exp(X_k)\exp(X_k')$ for $X_k \in N_0$ and $X_k' \in N_0'$. We need to investigate further the properties of X_k and X_k'. Firstly, we know that $h_k \notin \exp(W_0)$, so since $X_k \in N_0 \subset W_0$, we deduce that $X_k' \neq 0$ for all k. Secondly, since $X_k \in N_0 \subset W_0 \subset \mathfrak{h}$, we know that $\exp(X_k) \in H$. Thus it follows from $\exp(X_k)\exp(X_k') = h_k$ that $\exp(X_k') = \exp(-X_k)h_k \in H$, that is $e \neq \exp(X_k') \in H \cap \exp(N_0' \smallsetminus \{0\})$. We will show that this is impossible. In fact, let us consider the sequence $(X'_k)_{k\geq 1}$ constructed above and let $L_k := \mathbb{R}X'_k$, which is an element of the projective space $\mathbb{P}(\mathfrak{h}')$. Since $\mathbb{P}(\mathfrak{h}')$ is compact, up to passing to a subsequence, the sequence L_k converges, say to $L \in \mathbb{P}(\mathfrak{h}')$.



This means that if we take $X' \in L$ and $\varepsilon > 0$, then for any $k \geq k(\varepsilon)$ large enough we have:

- $L_k \cap B(X', \varepsilon) \neq \varnothing$;
- $\|X'_k\| < \varepsilon$;
- There exists $n_k \in \mathbb{Z}$ such that $\|X' - n_k X'_k\| < \varepsilon$, that is $n_k X'_k \to X'$. This last property follows from the Archimedean property of real numbers.

But then

$$\exp X' = \lim_{k\to\infty} \exp(n_k X'_k) = \lim_{k\to\infty} \exp(X'_k)^{n_k} \in H.$$

This is a contradiction, since $X' \in L \subset \mathbb{P}(\mathfrak{h}')$, and so $\exp X' \notin H$. $\square$

3.6 The Adjoint Representation

Let G be a Lie group. A *representation* of G over k (where $k = \mathbb{R}$ or $\mathbb{C}$) is a continuous (hence smooth by Theorem 3.14) homomorphism $\pi : G \mapsto \mathrm{GL}(n, k)$. A *representation* of a Lie algebra $\mathfrak{g}$ over k is a homomorphism $\mathfrak{g} \to \mathfrak{gl}(n, k)$. Any Lie group representation π gives in turn by differentiation a Lie algebra representation $d_e\pi : \mathfrak{g} \mapsto \mathfrak{gl}(n, k)$. Both G and $\mathfrak{g}$ act on k^n (via π and $d_e\pi$ respectively) and it is easy to see that if $V \subset k^n$ and G is connected, then V is $\pi(G)$-invariant if and only if it is $d_e\pi(\mathfrak{g})$-invariant. In fact, in the appropriate coordinates, the stabilizer $H_V \leq \mathrm{GL}(n, k)$ of V takes the form $H_V = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and likewise for $\mathrm{Lie}(H_V) \subset \mathfrak{gl}(n, \mathbb{R})$. Thus

$\pi(G) \subset H_V$ if and only if $d_e\pi(\mathfrak{g}) \subset \mathrm{Lie}(H_V)$.

For each $g \in G$ we consider the inner automorphism of G defined in (2.6), $c_g(h) := ghg^{-1}$. Since $c_g(e) = e$ for all $g \in G$, $d_e c_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, and since $c_g \circ c_h = c_{gh}$, it defines a representation of G into $\mathrm{GL}(\mathfrak{g})$.

Definition 3.19. Adjoint representation

Let G be a Lie group.

1. *The* adjoint representation *of G is*

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto d_e c_g\,. \end{aligned}$$

2. *The adjoint representation of the Lie algebra $\mathfrak{g}$*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined by

$$\mathrm{ad} = d_e \mathrm{Ad}\,.$$

♣

Remark By the naturality of the exponential map, we have the commutativity of the following diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp_G"'] & \mathfrak{g} \arrow[d, "\exp_G"] \\ G \arrow[r, "c_g"] & G \end{tikzcd}$$

that is

$$\exp_G \circ \mathrm{Ad}(g) = c_g \circ \exp_G\,, \tag{3.9}$$

as well as of the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp_G"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp_{\mathrm{GL}(\mathfrak{g})}"] \\ G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \end{tikzcd}$$

that is

$$\exp_{\mathrm{GL}(\mathfrak{g})} \circ \mathrm{ad}(X) = \mathrm{Ad} \circ \exp_G(X) \tag{3.10}$$

for all $X \in \mathfrak{g}$.

Proposition 3.14

If G is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$, then

$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

for all $g \in G$ and all $X \in \mathfrak{g}$.

Proof Since $G \leq \mathrm{GL}(n, \mathbb{R})$, the assertion follows from the fact that c_g is linear, and so $d_e c_g = c_g$ and from (3.9). □

One can also give an explicit characterization of ad, which holds however for all Lie groups and not necessarily only for the linear ones.

Proposition 3.15

If G is a Lie group with Lie algebra $\mathfrak{g}$, then for all $X, Y \in \mathfrak{g}$

$$\mathrm{ad}(X)(Y) = [X, Y] \, .$$

Proof We start by illustrating a proof that is just very sloppy but gives the idea of what one is looking for. By definition of Ad and ad, if $X, Y \in \mathfrak{g}$ we can write

$$\begin{aligned}
\mathrm{ad}(X)(Y) &= (d_e \mathrm{Ad})(X)(Y) \\
&= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX))(Y) \\
&= \frac{d}{dt}\Big|_{t=0} (d_e c_{\exp(tX)})(Y) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} c_{\exp(tX)}(\exp(sY)) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \exp(tX)\exp(sY)\exp(-tX)
\end{aligned}$$

If G is a linear group, $d_e c_g = c_g$, so that $\mathrm{Ad}(g) = c_g$ (Proposition 3.14) and

$$\mathrm{ad}(X)(Y) = \frac{d}{dt}\Big|_{t=0} \exp(tX) Y \exp(-tX) = XY - YX \, .$$

If G is not linear one would need to compute also the derivative with respect to s.

The problem (in fact, one of the problems) with the above "proof" is that we considered only vector fields at the identity, while instead if we want to take derivatives we need to consider the vector fields defined in a neighborhood of e. In other words, if $Z \in \mathfrak{g}$, we denote by $\widetilde{Z} \in \mathrm{Vect}(G)^G$ the left invariant vector field whose value at $e \in G$ is Z.

The naturality of the exponential map (3.10), gives for all $tX \in \mathfrak{g}$

$$\mathrm{Ad}(\exp_G(tX)) = \exp_{\mathrm{GL}(\mathfrak{g})}(\mathrm{ad}(tX))\,.$$

But we saw in Corollary 3.8 that $\exp_{\mathrm{GL}(\mathfrak{g})}$ is just the matrix exponential, so that

$$\begin{aligned}\mathrm{Ad}(\exp(tX))(Y) &= \exp_{\mathrm{GL}(\mathfrak{g})}(\mathrm{ad}(tX))(Y) = e^{\mathrm{ad}(tX)}Y = \\ &= Y + t\,\mathrm{ad}(X)Y + \frac{t^2}{2}R(t,X)Y\,.\end{aligned} \tag{3.11}$$

where $R(t,X)$ is the smooth remainder. By considering the invariant vector fields associated to the ones in (3.11) and differentiating one obtains for $f \in C^\infty(G)$

$$(\mathrm{ad}(X)Y\widetilde{)}_g(f) = \frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}(\exp(tX)(Y)\widetilde{)}_g(f) \tag{3.12}$$

and the rest of the proof will be the technical calculation of the derivative on the right hand side.

Recall that $\tilde{Z}_g(f)$ is the directional derivative of f at g in the direction of $\tilde{Z}_g$, that is

$$\tilde{Z}_g(f) = (d_gf)(\tilde{Z}_g) = (d_gf)(d_eL_g)(Z) = d_e(f\circ L_g)(Z).$$

1. Since Z is the tangent vector at $s=0$ to the curve $s\mapsto \exp(sZ)$, we have at the point $g\exp(tX)$,
$$\begin{aligned}\tilde{Z}_{g\exp(tX)}(f) &= d_e(f\circ L_{g\exp(tX)})(Z) = \frac{d}{ds}\Big|_{s=0}(f\circ L_{g\exp(tX)})(\exp(sZ)) \\ &= \frac{d}{ds}\Big|_{s=0} f(g\exp(tX)\exp(sZ))\,.\end{aligned}$$
2. If instead $Z = \mathrm{Ad}(h)(Y) = d_ec_h(Y)$ is the tangent vector to the curve $s\mapsto h\exp(sY)h^{-1}$, then
$$\begin{aligned}(\mathrm{Ad}(h)(Y)\widetilde{)}_g(f) &= d_e(f\circ L_g)(\mathrm{Ad}(h)(Y)) = \frac{d}{ds}\Big|_{s=0}(f\circ L_g)(h\exp(sY)h^{-1}) = \\ &= \frac{d}{ds}\Big|_{s=0} f(gh\exp(sY)h^{-1}).\end{aligned}$$
3. Finally, we use that if $F(u,v)$ is a differentiable function, then
$$\frac{d}{dt}\Big|_{t=0}F(t,t) = \frac{d}{dt}\Big|_{t=0}F(t,0) + \frac{d}{dt}\Big|_{t=0}F(0,t).$$
Indeed, let $\Delta\colon \mathbb{R}\to\mathbb{R}^2$ be the diagonal embedding $t\mapsto(t,t)$, so that $F(t,t) = F\circ\Delta(t)$.

Then

$$\left.\frac{d}{dt}\right|_{t=0} F\circ\Delta(t) = d_0(F\circ\Delta)(1) = d_{(0,0)}F\circ d_0\Delta(1) = d_{(0,0)}F(1,1)$$
$$= \frac{\partial F}{\partial u}(0,0) + \frac{\partial F}{\partial v}(0,0) = \left.\frac{d}{dt}\right|_{t=0} F(t,0) + \left.\frac{d}{dt}\right|_{t=0} F(0,t).$$

We can now finally complete the proof.

We need to define very-widetilde command

From (3.12) and 2. with $h=\exp(tX)$ one has

$$\widetilde{(\mathrm{ad}(X)(Y))}_g(f) = \left.\frac{d}{dt}\right|_{t=0}\left.\frac{d}{ds}\right|_{s=0} f(g\exp(tX)\exp(sY)\exp(-tX))$$
$$\overset{3.}{=} \left(\left.\frac{d}{dt}\right|_{t=0}\left.\frac{d}{ds}\right|_{s=0} f(g\exp(tX)\exp(sY))\right)$$
$$+\left(\left.\frac{d}{dt}\right|_{t=0}\left.\frac{d}{ds}\right|_{s=0} f(g\exp(sY)\exp(-tX))\right)$$
$$\overset{1.}{=} \left.\frac{d}{dt}\right|_{t=0} \tilde{Y}_{g\exp(tX)}(f) - \left.\frac{d}{ds}\right|_{s=0} \tilde{X}_{g\exp(sY)}(f)$$
$$\overset{1.}{=} (\tilde{X}\circ\tilde{Y})_g(f) - (\tilde{Y}\circ\tilde{X})_g(f) = [\tilde{X},\tilde{Y}]_g(f);$$

where in the next to the last equality 1. was applied to $\tilde{Z}_g = \tilde{Y}_{g\exp(tX)}$ and $\tilde{Z}_g = \tilde{X}_{g\exp(sY)}$.

Evaluating at $g=e$, we conclude that $\mathrm{ad}(X)(Y) = [X,Y]$. □

Example 3.32 Example of $\mathrm{Ad}(a)$ and $\mathrm{ad}(a)$ for $a = \mathrm{diag}(\lambda_1,\dots,\lambda_n) \in \mathrm{GL}(n,\mathbb{R})$ and of $\mathrm{Ad}(\mathrm{SL}(2,\mathbb{R}))$ and $\mathrm{ad}(\mathfrak{sl}(2,\mathbb{R}))$.

Remark A priori we only have that $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, but in fact one can easily check using the Jacobi identity that

$$\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})\,.$$

In fact, recall that if $\mathfrak{g}$ is a Lie algebra, a derivation $\delta\in\mathrm{Der}(\mathfrak{g})$ is an endomorphism $\delta\colon\mathfrak{g}\to\mathfrak{g}$ such that $\delta[X,Y] = [\delta(X),Y] + [X,\delta(Y)]$. Then using the Jacobi identity,

$$\mathrm{ad}(X)[Y,Z] \quad = [X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] = [\mathrm{ad}(X)Y,Z] + [Y,\mathrm{ad}(X)Z]\,.$$

For further structural properties of the group of automorphisms, see [8, P. 159].

Definition 3.20. Ideal in a Lie algebra

A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is an ideal *if for all $X\in\mathfrak{h}$ and all $Y\in\mathfrak{g}$, $[X,Y]\in\mathfrak{h}$.* ♣

Proposition 3.16

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, H a closed connected subgroup with Lie algebra $\mathfrak{h}$. Then H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Proof

$$H \text{ is normal } \Leftrightarrow c_g(H) = H \text{ for all } g \in G \Leftrightarrow \mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h} \text{ for all } g \in G \Leftrightarrow$$
$$\Leftrightarrow \mathrm{ad}(X)(\mathfrak{h}) \subset \mathfrak{h} \text{ for all } X \in \mathfrak{g} \Leftrightarrow \mathfrak{h} \text{ is an ideal.}$$

□

Proposition 3.17

Let G be a connected Lie group. Then $Z(G) = \ker \mathrm{Ad}$ and $Z(\mathfrak{g}) = \ker \mathrm{ad}$.

Proof We will show the statement for Lie groups, and the one for Lie algebras will follow readily.

If $g \in Z(G)$, then $c_g = \mathrm{Id}$, so that $\mathrm{Ad}(g) = \mathrm{Id}$, which means that $g \in \ker \mathrm{Ad}$.

Conversely, let us suppose that $g \in \ker \mathrm{Ad}$. Then $d_e c_g(X) = \mathrm{Ad}(g)(X) = X$ for all $X \in \mathfrak{g}$. Applying $\exp$ on the last two terms of the above equality and using (3.9), one obtains that $c_g \exp X = \exp X$. Thus c_g commutes with all elements that are in the image of the exponential map. However the same proof still works for all elements in G, since G can be generated by elements contained in a neighbourhood where $\exp$ is a diffeomorphism (see the remark after Cartan's Theorem 3.11). □

Chapter 3 Exercise

1. Let G be a Lie group and $H \trianglelefteq G$ a closed normal subgroup. Show that G/H is a Lie group and that $\mathrm{Lie}(G/H) = \mathrm{Lie}(G)/\mathrm{Lie}(H)$.
2. Show that $d_I \det = \mathrm{tr}$, the usual trace map $\mathbb{R}^{n\times n} \to \mathbb{R}$. Deduce that $SL(n, \mathbb{R})$ is a Lie group of dimension $(n^2 - 1)$.
3. Show that $\mathrm{O}(p, q)$ and $\mathrm{U}(p, q)$ are Lie groups.
4. If $X_i \in \mathrm{Vect}(M)$ is φ-related to $X_i' \in \mathrm{Vect}(M')$, for $i = 1, 2$, then $[X_1, X_2]$ is φ-related to $[X_1', X_2']$.
5. Let M, M be smooth manifolds, let $i\colon N \to M$ be an immersed submanifold and let $f\colon\ : M \to M$ be a smooth map such that $f(M) \subseteq N$. Show that $i^{-1} \circ f\colon M \to N$ is smooth.
6. Let G be a connected Lie group, H a topological group and $p : H \to G$ a covering map. Then there exists a unique Lie group structure on H such that p is a Lie group homomorphism

and the kernel of p is a discrete subgroup of H.

7. Let G be a connected Lie group and (H, p) a covering with the Lie group structure given by the previous point. Then p is a local isomorphism of Lie groups, and $d_e p$ is an isomorphism of Lie algebras.
8. Let $p\colon H \to G$ be a connected Lie group homomorphism. Then p is a covering map if and only if $d_e p$ is an isomorphism.
9. Let $D \leq \mathbb{R}^n$ be a discrete subgroup. Then there exist $x_1, \ldots, x_k \in D$ such that:
   (a). $x_1, \ldots, x_k$ are linearly independent over $\mathbb{R}$;
   (b). D is the $\mathbb{Z}$-span of $x_1, \ldots, x_k$, that is $D = \mathbb{Z}x_1 + \cdots \mathbb{Z}x_k$ (in other words, $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-submodule).

   Thus a discrete subgroup of $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq n$.
10. Let H be an abstract subgroup of the Lie group G and let $\mathfrak{h}$ be a subspace of $\mathfrak{g} = \mathrm{Lie}(G)$. Let $0 \in U_0 \subset \mathfrak{g}$ and $e \in V_e \subset G$ be open neighborhoods such that $\exp : U_0 \to V_e$ is a diffeomorphism. Suppose that $\exp(U_0 \cap \mathfrak{h}) = V_e \cap H$. Then:
    (a). H is a Lie subgroup of G with the induced topology;
    (b). $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;
    (c). $\mathfrak{h} = \mathrm{Lie}(H)$.
11. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then for t small enough

$$\exp(tX)\exp(tY) = \exp\left[t(X+Y) + O(t^2)\right],$$

where $\frac{1}{t^2}O(t^2)$ is bounded at $t = 0$.

Chapter 4 Structure Theory

4.1 Solvability

We defined in the previous section the adjoint representation of the Lie algebra of a Lie group. One can also define the adjoint representation of an abstract Lie algebra $\mathfrak{g}$.

Definition 4.1. Adjoint representation, II

Let $\mathfrak{g}$ be a Lie algebra. The adjoint representation *is defined as*

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto \mathrm{ad}(X)\,, \end{aligned}$$

where $\mathrm{ad}(X)(Y) := [X, Y]$ *for all* $Y \in \mathfrak{g}$.

♣

That this definition coincides with the one for the Lie algebra of a Lie group is the content of Proposition 3.15.

Definition 4.2. Characteristic ideal

Let $\mathfrak{g}$ be a Lie algebra. An ideal $\mathfrak{h}$ is characteristic *if* $\delta(\mathfrak{h}) \subset \mathfrak{h}$ *for every derivation* $\delta \in \mathrm{Der}(\mathfrak{g})$.

♣

The importance of characteristic ideals lies in the following result:

Lemma 4.1

If $\mathfrak{k} \subset \mathfrak{g}$ is an ideal and $\mathfrak{h} \subset \mathfrak{k}$ is a characteristic ideal in $\mathfrak{k}$, then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

♡

Proof We saw that because of the Jacobi identity, if $X \in \mathfrak{g}$ the endomorphism $\delta_X\colon \mathfrak{g} \to \mathfrak{g}$ defined by $\delta_X(Y) := [X, Y]$ is a derivation of $\mathfrak{g}$. Since $\mathfrak{k} \subset \mathfrak{g}$ is an ideal, $\delta_X(\mathfrak{k}) \subset \mathfrak{k}$ and hence $\delta_X \in \mathrm{Der}(\mathfrak{k})$. Since $\mathfrak{h} \subset \mathfrak{k}$ is characteristic, $\delta_X(Y) = [X, Y] \in \mathfrak{h}$ for all $Y \in \mathfrak{h}$. Thus $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. □

Example 4.1 If $\delta \in \mathrm{Der}(\mathfrak{g})$, then $\delta[X, Y] = [\delta X, Y] + [X, \delta Y]$. So $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal, where $[\mathfrak{g}, \mathfrak{g}]$ is defined as the span of elements of the form $[X, Y]$, for $X, Y \in \mathfrak{g}$.

We set inductively

$$\mathfrak{g}^{(1)} :=[\mathfrak{g}, \mathfrak{g}]$$
$$\mathfrak{g}^{(i+1)} :=[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]\,.$$

Then $\mathfrak{g}^{(i+1)}$ is a characteristic ideal of $\mathfrak{g}^{(i)}$, hence an ideal of $\mathfrak{g}$, by Lemma 4.1.

Definition 4.3. Solvable Lie algebra

Let $\mathfrak{g}$ be a Lie algebra. We call

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$$

the derived series *of $\mathfrak{g}$. We say that $\mathfrak{g}$ is* solvable *if $\mathfrak{g}^{(k)} = \{0\}$ for some k. The smallest k for which $\mathfrak{g}^{(k)} = 0$ is the* length *of $\mathfrak{g}$.* ♣

The simplest example of a solvable Lie algebra is an Abelian one.

Proposition 4.1

Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:

1. *$\mathfrak{g}$ is solvable.*
2. *There exists a chain of subalgebras $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n = \{0\}$ such that*
   - *(a). $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$;*
   - *(b). $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is Abelian.*

♠

Proof (1. $\Rightarrow$ 2.) Set $\mathfrak{g}_i := \mathfrak{g}^{(i)}$. Then the $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1} = \mathfrak{g}^{(i)}/[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$ is Abelian.

(2. $\Rightarrow$ 1.) We will argue by induction on the length n of the series to show that $\mathfrak{g}^{(k)} \subset \mathfrak{g}_k$, so that if $\mathfrak{g}_k = \{0\}$ for some k then also $\mathfrak{g}^{(k)} = \{0\}$.

If $n = 1$ then $\mathfrak{g} \supset \mathfrak{g}_1 = \{0\}$, so that $\mathfrak{g}$ is Abelian and hence solvable.

Now let $n > 1$ and let us suppose that $\mathfrak{g}^{(n-1)} \subset \mathfrak{g}_{n-1}$. Then

$$\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \subset [\mathfrak{g}_{n-1}, \mathfrak{g}_{n-1}] \subset \mathfrak{g}_n\,,$$

where the last inclusion follows from the fact that $\mathfrak{g}_{n-1}/\mathfrak{g}_n$ is Abelian. □

Corollary 4.1

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable. If $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable of length respectively n and k, then $\mathfrak{g}$ is of length

$\leq n + k.$

Remark It follows that the class of solvable Lie algebras is the smallest class $\mathcal{C}$ such that

1. Abelian Lie algebras are in $\mathcal{C}$;
2. If $\{0\} \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a short exact sequence and $\mathfrak{h}, \mathfrak{g}/\mathfrak{h} \in \mathcal{C}$, then $\mathfrak{g} \in \mathcal{C}$.

Proof [Proof of Corollary 4.1] $(\Rightarrow)$ Obvious.

$(\Leftarrow)$ Let $\mathfrak{g}/\mathfrak{h} \supset \mathfrak{l}_1 \supset \mathfrak{l}_2 \supset \cdots \supset \mathfrak{l}_k = \{0\}$ be a chain of ideals such that $\mathfrak{l}_i/\mathfrak{l}_{i+1}$ is Abelian and similarly $\mathfrak{h} \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_n = \{0\}$. Let $p \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the quotient map. Then defining $\mathfrak{g}_i := p^{-1}(\mathfrak{l}_i)$ for $1 \leq i \leq k$ and $\mathfrak{g}_i := \mathfrak{h}_{i-k}$ for $k < i \leq k+n$ we obtain a chain of ideals

$$\mathfrak{g} \supset p^{-1}(\mathfrak{l}_1) \supset \cdots \supset p^{-1}(\mathfrak{l}_k) = \mathfrak{h} \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_n = \{0\}$$

such that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is Abelian. □

Example 4.2

$$\mathfrak{n} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{R}) \right\}; \mathfrak{n}^{(1)} = \left\{ \begin{pmatrix} 0 & * & & * \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & 0 \end{pmatrix} \right\}; \mathfrak{n}^{(2)} = \left\{ \begin{pmatrix} 0 & 0 & * & & * \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & * \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{pmatrix} \right\};$$

and so on. So $\mathfrak{n}$ is a solvable Lie algebra.

Definition 4.4. Solvable Lie group

Let G be a connected Lie group. Then G is solvable *if* $\mathrm{Lie}(G)$ *is solvable.*

Example 4.3 The group

$$N := \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathrm{GL}(n, \mathbb{R}) \right\}$$

is solvable.

Proposition 4.2

Let G be a connected Lie group G. The following are equivalent:

1. *G is solvable.*
2. *There exists a sequence of closed and connected subgroups $\{G_i\}$ such that:*

(a). $G \geq G_1 \geq \cdots \geq G_k = \{e\}$;

(b). $G_{i+1} \trianglelefteq G_i$;

(c). G_i/G_{i+1} *is Abelian.*

Proof (2. $\Rightarrow$ 1.) Obvious, by taking the Lie algebras of these closed Lie subgroups (see Exercise 1.) and using Proposition 4.1.

(1. $\Rightarrow$ 2.) If G is solvable, then $\mathfrak{g}$ is solvable, so that the derived series

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(k)} = \{0\} \tag{4.1}$$

terminates. The sequence

$$G \geq G_1 \geq G_2 \geq \cdots \geq G_k = \{e\} \tag{4.2}$$

of Lie groups G_j such that $\mathrm{Lie}(G_j) = \mathfrak{g}^{(j)}$ satisfies the required properties, with the only exception that the G_j are not necessarily closed. We will argue by induction on the length of the series.

If $k = 1$, there is nothing to show. Let us assume now the assertion for all $r \leq k-1$. The Lie algebra $\mathfrak{g}^{(k-1)}$ in (4.1) is Abelian and so is the group $G_{k-1} \trianglelefteq G$ in (4.2). Its closure $\overline{G_{k-1}}$ is also an Abelian normal subgroup in G with Lie algebra $\mathfrak{g}_{k-1} := \mathrm{Lie}(\overline{G_{k-1}})$ which is an Abelian ideal in $\mathfrak{g}$. Moreover $\mathfrak{g}/\mathfrak{g}_{k-1} = \mathrm{Lie}(G/\overline{G_{k-1}})$ is solvable of length $\leq k-1$. By inductive hypothesis there exist closed connected subgroups

$$H := G/\overline{G_{k-1}} \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_{k-1} = \{e\}$$

such that H_j/H_{j+1} is Abelian for all $0 \leq j \leq k-2$. If $\pi : G \to G/\overline{G_{k-1}}$, let

$$\begin{aligned} \widetilde{G}_j &:= \pi^{-1}(H_j), \qquad \text{for } 0 \leq j \leq k-2 \\ \widetilde{G}_{k-1} &:= \overline{G_{k-1}}\,. \end{aligned}$$

Then $\widetilde{G}_j$ are all closed connected normal subgroups in G. Since $[H_j, H_j] < H_{j+1}$, then $[\widetilde{G}_j, \widetilde{G}_j] < \widetilde{G}_{j+1}$ and

$$G \geq \widetilde{G}_1 \geq \widetilde{G}_2 \geq \cdots \geq \widetilde{G}_{k-1} \geq \widetilde{G}_k = \{e\}$$

is the required series.

□

Theorem 4.1. Lie's Theorem

1. *Let G be a connected solvable Lie group and let $\pi\colon G \to \mathrm{GL}(n,\mathbb{C})$ be a complex representation. Then there is a basis of $\mathbb{C}^n$ with respect to which $\pi(G)$ consists of* *upper triangular matrices, that is* $\pi(G) \leq \left\{ \begin{pmatrix} * & & * \\ 0 & \ddots & \\ 0 & & * \end{pmatrix} \in \mathrm{GL}(n, \mathbb{C}) \right\}$.

2. *Let* $\mathfrak{g}$ *be a solvable Lie algebra and* $\rho \colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ *a complex representation. Then there is a basis of* $\mathbb{C}^n$ *with respect to which* $\rho(\mathfrak{g})$ *consists of upper triangular matrices.* ♡

Corollary 4.2. Solvable matrix groups

Let V *be an* n*-dimensional* $\mathbb{C}$*-vector space and let* $G < \mathrm{GL}(V)$ *a Lie group with Lie algebra* $\mathfrak{g} \subset \mathfrak{gl}(V)$. *Then the following are equivalent:*

1. *The Lie group* G *(or the Lie algebra* $\mathfrak{g}$*) is solvable;*
2. *There exists a* G*-invariant (or* $\mathfrak{g}$*-invariant) sequence of* $\mathbb{C}$*-subspaces*
$$V \supset V_{n-1} \supset V_{n-2} \supset \cdots \supset V_1 \supset \{0\}$$
such that $\dim V_j = j$, *where* $n = \dim V$;
3. *There exists a basis of* V *such that* G *(or* $\mathfrak{g}$*) consists of upper-triangular matrices.* ♡

We now move to the proof of the Lie Theorem, for which we will need some preliminaries.

Definition 4.5

1. *Let* G *be a connected Lie group and* $\pi \colon G \to \mathrm{GL}(n, \mathbb{C})$ *a representations. We say that* v *is a* common eigenvector of $\{\pi(g) : g \in G\}$ *if* $\pi(g)v = \chi(g)v$, *where* $\chi \colon G \to \mathbb{C}^*$ *is a smooth homomorphism.*
2. *Let* $\rho \colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ *be a Lie algebra representation. We say that* v *is a* common eigenvector of $\{\rho(X) : X \in \mathfrak{g}\}$ *if* $\rho(X)v = \lambda(X)v$, *where* $\lambda \in \mathfrak{g}^*$ *is a linear map.* ♣

Lemma 4.2

Let G *be a connected Lie group,* V *a complex vector space and* $\pi \colon G \to \mathrm{GL}(V)$ *be a Lie group representation. A vector* $v \in V$ *is a common eigenvector of* $\{\pi(g) : g \in G\}$ *if and only if it is a common eigenvector of* $\{d_e\pi(X) : X \in \mathfrak{g}\}$. *Moreover*
$$\chi(\exp(X)) = e^{\lambda(X)}$$
for all $X \in \mathfrak{g}$. ♡

Proof By differentiating, it is obvious that if v is a common eigenvector for $\pi(G)$ it is also a common eigenvector for $d_e\pi(\mathfrak{g})$.

To see the converse, let v be a common eigenvector of $d_e\pi(\mathfrak{g})$ and let $G_v := \{g \in G :$

$\pi(g)\mathbb{C}v = \mathbb{C}v\}$ be the stabilizer of the line $\mathbb{C}v$. We want to show that $G_v = G$. By definition G_v is a closed subgroup of G and hence a Lie group whose Lie algebra is

$$\begin{aligned}\mathrm{Lie}(G_v) &= \{X \in \mathfrak{g} : \exp_G(tX) \in G_v \text{ for all } t \in \mathbb{R}\}\\ &= \{X \in \mathfrak{g} :\ \pi(\exp_G(tX))\mathbb{C}v = \mathbb{C}v \text{ for all } t \in \mathbb{R}\}\\ &= \{X \in \mathfrak{g} :\ \exp_{\mathrm{GL}(V)}(t\, d_e\pi(X))\mathbb{C}v = \mathbb{C}v \text{ for all } t \in \mathbb{R}\}\,.\end{aligned}$$

Now observe that if $A \in \mathrm{End}(V)$, then

$$\exp_{\mathrm{GL}(V)}(tA)\mathbb{C}v = \mathbb{C}v \Leftrightarrow A(\mathbb{C}v) \subset \mathbb{C}v\,.$$

In fact $(\Leftarrow)$ is immediate by the exponential series and $(\Rightarrow)$ follows from the fact that $A = \lim_{t\to 0} \frac{\exp_{\mathrm{GL}(V)}(tA)-\mathrm{Id}}{t}$.

Thus

$$\mathrm{Lie}(G_v) = \{X \in \mathfrak{g} :\ d_e\pi(X)(\mathbb{C}v) \subset \mathbb{C}v\} = \mathfrak{g}$$

by hypothesis. Since G s connected, this implies that $G_v = G$. Thus for all $g \in G$ there is a well defined $\chi(g) \in \mathbb{C}^*$ with $\pi(g)v = \chi(g)v$ and since $g \mapsto \pi(g)v$ is smooth, so is χ. Finally,

$$\chi(\exp_G(X))v = \pi(\exp_G(X))v = \exp_{\mathrm{GL}(V)}(d_e\pi(X))v = e^{\lambda(X)}v\,. \qquad \square$$

Remark Suppose that $H \trianglelefteq G$ is a connected normal subgroup. Then G acts on H by conjugation $G \times H \to H$, $(g,h) \mapsto c_g(h) = ghg^{-1}$, hence it acts on $\mathrm{Hom}(H,\mathbb{C}^*)$ via $(g,\chi) \mapsto g \cdot \chi$, where $g \cdot \chi(h) := \chi(c_g^{-1}(h)) = \chi(g^{-1}hg)$. Observe that $\mathrm{Hom}(H,\mathbb{C}^*)$ can be given a topology (e.g. the topology of uniform convergence on compact sets, or the topology of pointwise convergence) such that the G-action is continuous. Then if $\chi \in \mathrm{Hom}(H,\mathbb{C}^*)$ is such that the G-orbit of χ is finite, it follows that χ is a fixed point. In fact, if χ were not a fixed point, then the G-orbit would be discrete and not reduced to a single point, which is not possible since G is connected and the G-action is continuous.

In a solvable group, connected normal subgroups always exist, as the following lemma shows.

Lemma 4.3

If G is solvable and $\dim G \geq 1$, there exists a closed connected non-trivial normal subgroup $H \trianglelefteq G$ of codimension 1. ♡

Proof Since G is solvable, there exists a closed connected normal subgroup $G_1 \trianglelefteq G$ such that G/G_1 is Abelian, so that $G/G_1 \cong \mathbb{R}^n \times \mathbb{T}^k$. Let us choose a closed codimension 1 subgroup $H_1 < G/G_1$: if this were not possible it would mean that G/G_1 has dimension 1, hence G_1 is a codimension 1 normal subgroup of G and we would be done. Then $H := p^{-1}(H_1) < G$, where $p\colon G \to G/G_1$ is the projection, is a closed connected subgroup that is normal, since G/G_1 is Abelian, and has codimension 1. In fact, $H_1 \lhd G/G_1 \Rightarrow p^{-1}(H_1) \lhd G$ and a dimension count on the tangent spaces of the underlying manifolds gives the desired assertion. □

Corollary 4.3

If $\mathfrak{g}$ is a solvable Lie algebra there exists an ideal $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1. ♡

We can now prove Lie's Theorem 4.1 for Lie groups. The statement for Lie algebras is very similar and is left as an exercise (Exercise **??**).

Proof We will prove that there exists a common eigenvector $v \in \mathbb{C}^n$. Then we can iterate the proof, by considering a representation on $\mathbb{C}^n/\mathbb{C}v$.

The proof will be by induction on $\dim G$. If $\dim G = 1$, then this is just the fact that every complex matrix has an eigenvalue, together with Lemma 4.2.

Now suppose that $\dim G > 1$. By Lemma 4.3 there exists a closed connected non-trivial normal subgroup $H \trianglelefteq G$ of codimension 1. For each $\chi \in \mathrm{Hom}(H, \mathbb{C}^*)$ we set

$$V_\chi := \{v \in \mathbb{C}^n : \pi(h)v = \chi(h)v \text{ for all } h \in H\}\,.$$

By inductive hypothesis H has a common eigenvector, that is there exists $\chi \in \mathrm{Hom}(H, \mathbb{C}^*)$ such that $V_\chi \neq 0$. We now claim that if $\chi_1, \dots, \chi_\ell$ are distinct characters of H such that $V_{\chi_1} \neq \{0\}, \dots, V_{\chi_\ell} \neq \{0\}$, then the sum $V_{\chi_1} + \cdots + V_{\chi_\ell}$ is direct. By contradiction take ℓ minimal for which this fails, so that $\ell \geq 2$. Let $v_i \in V_{\chi_i}$, $v_i \neq 0$, $1 \leq i \leq \ell$ such that

$$\sum_{i=1}^{\ell} v_i = 0\,, \tag{4.3}$$

and, $\chi_{\ell-1} \neq \chi_\ell$, let h with $\chi_{\ell-1}(h) \neq \chi_\ell(h)$. Applying h to (4.3), dividing by $\chi_\ell(h)$ and subtracting from (4.3), we get

$$\sum_{i=1}^{\ell-1} \left[1 - \frac{\chi_i(h)}{\chi_\ell(h)}\right] v_i = 0\,,$$

which, in view of the fact that $1 - \frac{\chi_{\ell-1}(h)}{\chi_\ell(h)} \neq 0$, gives a non-trivial linear relation contradicting the minimality of ℓ. Since $\mathbb{C}^n$ is finite dimensional, this implies that there is only a finite number of $\chi \in \mathrm{Hom}(H, \mathbb{C}^\times)$ such that $V_\chi \neq \{0\}$. Moreover, for all $g \in G$ we have $\pi(g)V_\chi = V_{g\cdot\chi}$. Indeed, let $v \in V_\chi$ and $g \in G$. Then for all $h \in H$:

$$\pi(h)\pi(g)v = \pi(g)\pi(g^{-1}hg)v = \pi(g)\pi(c_g^{-1}(h))v = \pi(g)\chi(c_g^{-1}(h))v = (g \cdot \chi)(h)\pi(g)v,$$

and so $\pi(g)v \in V_{g\cdot\chi}$.

Since there is a finite number of χ such that $V_\chi \neq 0$, arguing as in the remark above we infer that V_χ is G-invariant, hence $\mathfrak{g}$-invariant. Now let $X \in \mathfrak{g}$ so that $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathrm{Lie}(H)$ and consider the representation $d_e\pi : \mathfrak{g} \to \mathfrak{gl}(V_\chi)$. Then X acting on V_χ has an eigenvalue, and since each vector of V_χ is an eigenvector of $d_e\pi(Y)$ for all $Y \in \mathfrak{h}$, the eigenvector of X on V_χ will be a common eigenvector of $d_e\pi(\mathfrak{g})$ and hence of $\pi(G)$. □

As a corollary of the proof of Lie's Theorem we have the following:

Corollary 4.4

1. *Every finite dimensional irreducible complex representation of a connected solvable Lie group (or Lie algebra) is one dimensional.*
2. *Every finite dimensional irreducible real representation of a connected solvable Lie group (or Lie algebra) is at most 2-dimensional.*

♡

Definition 4.6. Complexification

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The complexification *of $\mathfrak{g}$ is the Lie algebra*

$$\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g} = \mathfrak{g} + i\mathfrak{g},$$

where the bracket is induced by the one on $\mathfrak{g}$.

♣

Note that if $\{X_1, \dots, X_n\}$ is a basis of $\mathfrak{g}$ over $\mathbb{R}$, then $\{1 \otimes X_1, \dots, 1 \otimes X_n\}$ is a basis of $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$. Thus $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{g}$, but $\dim_{\mathbb{R}} \mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathbb{C} \times \dim_{\mathbb{R}} \mathfrak{g} = 2\dim_{\mathbb{R}} \mathfrak{g}$.

Corollary 4.5

The Lie algebra $\mathfrak{g}$ is solvable if and only if $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$ is upper-triangular with respect to some basis $\{1 \otimes X_1, \dots, 1 \otimes X_n\}$, where $\{X_1, \dots, X_n\}$ is a basis of $\mathfrak{g}$.

♡

Proof ($\Rightarrow$) Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ be the complexification of $\mathfrak{g}$. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}^{\mathbb{C}}$ is solvable and, by Lie's Theorem $\mathrm{ad} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{gl}(\mathfrak{g}^{\mathbb{C}})$ is such that $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$ is upper triangular.

($\Leftarrow$) Any Lie algebra of upper triangular matrices is solvable, hence $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$ is solvable. Moreover, from $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}}) = \mathrm{ad}(\mathfrak{g}) + i\,\mathrm{ad}(\mathfrak{g})$ we deduce that $\mathrm{ad}(\mathfrak{g})$ is solvable, since it is a subalgebra of $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$.

We conclude from the short exact sequence

$$\{0\} \to Z(\mathfrak{g}) \to \mathfrak{g} \to \operatorname{ad}(\mathfrak{g}) \to \{0\}$$

that $\mathfrak{g}$ is solvable, since $Z(\mathfrak{g})$ is Abelian and thus solvable. $\square$

Remark It is not true that if $\mathfrak{g}$ is solvable then $\operatorname{ad}(\mathfrak{g})$ is upper-triangular.

Application We will show that there are Lie groups which do not have faithful representations, that is we will exhibit a Lie group G such that for all $\pi : G \to \mathrm{GL}(n, \mathbb{C})$ there exists $g \in G$, $g \neq e$, such that $\pi(g) = I$.

Let $N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ be the Heisenberg group, with center

$$H = Z(N) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z \in \mathbb{R} \right\}.$$

Let us consider

$$D = H \cap \mathrm{SL}(3, \mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} n \in \mathbb{Z} \right\}.$$

We will show that if $\pi \colon G \to \mathrm{GL}(n, \mathbb{C})$ is any representation, then $\pi(H/D) = \{\mathrm{Id}\}$, so that

$$G := N/D \cong \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, t \in S^1 \right\}$$

does not have faithful complex representations.

Claim 4.1.1. $\pi(H/D) \leq \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} =: L.$

Claim 4.1.2. *L cannot have non-trivial compact subgroups, hence $\pi(H/D) = \{\mathrm{Id}\}$.*

Proof [Proof of Claim 4.1.2] We show that if $K \leq L$ is a non-trivial compact subgroup, K can be conjugated into any neighborhood of $\mathrm{Id} \in \mathrm{GL}(n, \mathbb{C})$, contradicting the “no small subgroup property" of the Lie group L (Theorem 3.12). To this purpose, let $g = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in$ $\mathrm{GL}(n, \mathbb{C})$ with $0 < \lambda_1 < \cdots < \lambda_n$. Then if $i < j$:

$$\left(c_g \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right)_{ij} = \left(\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix} \right)_{ij} =$$

$$= \frac{\lambda_i}{\lambda_j} \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}_{ij} \Rightarrow$$

$$\Rightarrow \left(c_g^n \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right)_{ij} = \left(\frac{\lambda_i}{\lambda_j} \right)^n \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}_{ij} .$$

Now if $\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \in K$, then the entries are bounded. Since $\lambda_i/\lambda_j < 1$ we have $\left(\frac{\lambda_i}{\lambda_j}\right)^n \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}_{ij} \to 0$ uniformly, and so $c_g^n \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \to I$ uniformly.

Hence $c_g^n(K)$ is eventually contained in any neighbourhood of I. □

Proof [Proof of Claim 4.1.1] We want to show that $\pi(H/D) \leq L$, where

$$H/D = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R}/\mathbb{Z} \right\} .$$

To this purpose, if we set $\rho := d_e\pi$, it will be enough to show that

$$\rho(\mathrm{Lie}(H/D)) \subseteq \left\{ \begin{pmatrix} 0 & * & & * \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n, \mathbb{C}) ,$$

where $\mathrm{Lie}(H/D) = \mathrm{Lie}(H) =: \mathfrak{h}$ since D is discrete. Since $\mathfrak{h}$ is Abelian, a direct application of Lie's Theorem would imply that $\rho(\mathfrak{h})$ can be written in upper triangular form, which is not enough as we need it to be strictly upper triangular. We can however argue as follows. Since $N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$, then

$$\mathfrak{n} = \mathrm{Lie}(N) = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n, \mathbb{C}),$$

which is solvable. By Lie's Theorem $\rho(\mathfrak{n})$ is upper triangular and hence $[\rho(\mathfrak{n}), \rho(\mathfrak{n})]$ is strictly upper triangular. But $\mathfrak{h} = [\mathfrak{n}, \mathfrak{n}]$, so that $\rho(\mathfrak{h}) = \rho([\mathfrak{n}, \mathfrak{n}]) = [\rho(\mathfrak{n}), \rho(\mathfrak{n})]$ is strictly upper triangular, as needed. □

4.2 Nilpotency

We want to refine the notion of solvability. If $\mathfrak{g}$ is a Lie algebra, we set inductively

$$\begin{aligned} C^1(\mathfrak{g}) &:= [\mathfrak{g}, \mathfrak{g}] \\ C^{n+1}(\mathfrak{g}) &:= [\mathfrak{g}, C^n(\mathfrak{g})] = \mathrm{ad}(\mathfrak{g})(C^n(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})^n(\mathfrak{g}). \end{aligned}$$

Definition 4.7. Nilpotent Lie algebra

Let $\mathfrak{g}$ be a Lie algebra. We call

$$\mathfrak{g} \supset C^1(\mathfrak{g}) \supset C^2(\mathfrak{g}) \supset \cdots$$

the central series *of $\mathfrak{g}$. We say that $\mathfrak{g}$ is* nilpotent *if $C^n(\mathfrak{g}) = \{0\}$ for some n.* ♣

Remark

1. By definition $\mathfrak{g}^{(1)} = C^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Moreover $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \subset [\mathfrak{g}, C^1(\mathfrak{g})] = C^2(\mathfrak{g})$ and by induction

$$\mathfrak{g}^{(n)} \subset C^n(\mathfrak{g}).$$

   Hence any nilpotent Lie algebra is solvable. We will see that the converse is not true, but that $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent (Proposition 4.6).
2. Each $C^j(\mathfrak{g})$ is a characteristic ideal, and moreover $C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g})$ is Abelian. In fact $[C^j(\mathfrak{g}), C^j(\mathfrak{g})] \subset [\mathfrak{g}, C^j(\mathfrak{g})] = C^{j+1}(\mathfrak{g})$. This, however, is not the point of nilpotent Lie

algebras. The important fact is that

$$C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g}) \subseteq Z(\mathfrak{g}/C^{j+1}(\mathfrak{g})), \tag{4.4}$$

which is much stronger than being Abelian. In particular it follows from (4.4) that if $C^{n+1}(\mathfrak{g}) = \{0\}$, then $C^n(\mathfrak{g}) \subset Z(\mathfrak{g})$. Hence, for a solvable Lie algebra the last non-zero ideal in the derived series is Abelian, while for a nilpotent Lie algebra the last non-zero ideal in the central series is central. In particular a non-Abelian solvable Lie algebra with no center cannot be nilpotent.

Example 4.4 $\mathfrak{g} = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})\right\}$ is solvable but not nilpotent. In fact $[\mathfrak{g},\mathfrak{g}] = \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\} = \mathfrak{g}^{(1)} = C^1(\mathfrak{g})$ is Abelian but $C^2(\mathfrak{g}) = [\mathfrak{g}, C^1(\mathfrak{g})] = C^1(\mathfrak{g})$. This is not surprising since $Z(\mathfrak{g}) = \{0\}$.

Proposition 4.3

Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:

1. *$\mathfrak{g}$ is nilpotent.*
2. *There exists a chain of subalgebras $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n = \{0\}$ such that*
   (a). $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$;
   (b). $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$.
3. *There exists $p \in \mathbb{N}$ such that $\mathrm{ad}(X_1) \circ \cdots \circ \mathrm{ad}(X_p) = \{0\}$ for all $X_1, \ldots, X_p \in \mathfrak{g}$.*

Proof (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) The proof by induction is similar to the one for solvable Lie algebras. We have $\mathfrak{g} = \mathfrak{g}_0$. Then $C^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}_0] \overset{(b)}{\subset} \mathfrak{g}_1$, and inductively, if $C^k(\mathfrak{g}) \subset \mathfrak{g}_k$, then $C^{k+1}(\mathfrak{g}) = [\mathfrak{g}, C^k(\mathfrak{g})] \subset [\mathfrak{g}, \mathfrak{g}_k] \overset{(b)}{\subset} \mathfrak{g}_{k+1}$. Then $\mathfrak{g}_n = \{0\}$ implies $C^n(\mathfrak{g}) = \{0\}$.

(1) $\Leftrightarrow$ (3) This is obvious since $C^k(\mathfrak{g})$ is generated by elements of the form $\mathrm{ad}(X_1)\circ\cdots\mathrm{ad}(X_k)(Y)$ for $X_1, \ldots, X_k, Y \in \mathfrak{g}$. □

Example 4.5 If $\mathfrak{g} = \left\{\begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R})\right\}$, then $\mathfrak{g}$ is nilpotent.

We saw that for solvable Lie algebras, if $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and

$\mathfrak{g}/\mathfrak{h}$ are solvable. The analogous statement for nilpotent Lie algebras cannot be true, because if it were we could show that any solvable Lie algebra is nilpotent by induction on $\dim \mathfrak{g}$. In fact, let us assume that $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is one-dimensional then it is certainly nilpotent. If $\dim \mathfrak{g} > 1$, given any ideal $\mathfrak{h} \subset \mathfrak{g}$, both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable. Since their dimension is smaller than the dimension of $\mathfrak{g}$, they would be nilpotent by the inductive hypothesis and hence $\mathfrak{g}$ would be nilpotent. The correct statement instead is the following:

Proposition 4.4

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an ideal.

1. *If $\mathfrak{g}$ is nilpotent, then $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are nilpotent.*
2. *If $\mathfrak{g}/\mathfrak{h}$ is nilpotent, and $\mathfrak{h} \subset Z(\mathfrak{g})$, then $\mathfrak{g}$ is nilpotent.*

♠

In other words, it is not enough that $\mathfrak{h}$ is nilpotent, but we need the stronger property that $\mathfrak{h} \subset Z(\mathfrak{g})$.

Proof 1. is obvious from the definition.

2. If $\mathfrak{g}/\mathfrak{h}$ is nilpotent, let $\mathfrak{g}/\mathfrak{h} \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_n = \{0\}$ be a chain of subalgebras with the properties as in Proposition 4.3, namely

1. $\mathfrak{h}_{j+1} \subset \mathfrak{h}_j$ is an ideal and
2. $[\mathfrak{g}/\mathfrak{h}, \mathfrak{h}_j] \subset \mathfrak{h}_{j+1}$.

If $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the projection, which is a Lie algebra homomorphism and $\mathfrak{g} \supset p^{-1}(\mathfrak{h}_1) \supset \cdots \supset p^{-1}(\mathfrak{h}_n) = \mathfrak{h} \supset \{0\}$, it is easy to check that

1. $p^{-1}(\mathfrak{h}_{j+1}) \subset p^{-1}(\mathfrak{h}_j)$ is an ideal and
2. $[\mathfrak{g}, p^{-1}(\mathfrak{h}_j)] \subset p^{-1}(\mathfrak{h}_{j+1})$.

Since $\mathfrak{h} \subset Z(\mathfrak{g})$, then $\mathfrak{h}_{n+1} := [\mathfrak{g}, \mathfrak{h}] = \{0\}$, so that $\mathfrak{g}$ is nilpotent. □

Definition 4.8. Nilpotent Lie group

Let G be a connected Lie group. Then G is nilpotent *if* $\mathrm{Lie}(G)$ *is nilpotent.*

♣

Proposition 4.5

Let G be a connected Lie group. The following are equivalent:

1. *G is nilpotent.*
2. *There exists a sequence of closed connected normal subgroups $G_i \lhd G$ such that*

(a). $[G, G_i] < G_{i+1}$ *and*

(b). $G > G_1 > \cdots > G_n = \{e\}$.

3. *There exists a sequence of closed connected normal subgroups* $G_i \lhd G$ *such that*

   (a). $G_i/G_{i+1} < Z(G/G_{i+1})$ *and*

   (b). $G > G_1 > \cdots > G_n = \{e\}$.

Proof Similar to the solvable case presented in Proposition 4.2. □

We gave an example to show that if $\mathfrak{g}$ is solvable, then $\mathfrak{g}$ is not neessarily nilpotent. The next proposition shows something more precise.

Proposition 4.6

$\mathfrak{g}$ *is solvable if and only if* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Proof ($\Leftarrow$) Suppose that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Then $[\mathfrak{g}, \mathfrak{g}]$ is solvable. So $\mathfrak{g}$ is solvable, with the same derived series as the one for $[\mathfrak{g}, \mathfrak{g}]$ shifted by one.

($\Rightarrow$) We will prove this implication in three steps.

(i) If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is solvable, where V is a $\mathbb{C}$-vector space, then by Lie's Theorem $\mathfrak{g}$ is upper triangular. This implies that $[\mathfrak{g}, \mathfrak{g}]$ is strictly upper triangular, and in particular nilpotent.

(ii) If $\mathfrak{g}$ is a solvable complex Lie algebra but not necessarily $\mathfrak{g} \subset \mathfrak{gl}(V)$, by Lie's Theorem $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is upper triangular. So $[\mathrm{ad}(\mathfrak{g}), \mathrm{ad}(\mathfrak{g})] = \mathrm{ad}([\mathfrak{g}, \mathfrak{g}])$ is strictly upper triangular, hence nilpotent. To deduce that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, by Proposition 4.4 it suffices to show that $\ker(\mathrm{ad}|_{[\mathfrak{g},\mathfrak{g}]}) \subseteq Z([\mathfrak{g}, \mathfrak{g}])$. But this is immediate since $\ker(\mathrm{ad}) = Z(\mathfrak{g}) \subseteq Z([\mathfrak{g}, \mathfrak{g}])$.

(iii) If now $\mathfrak{g}$ is a real Lie algebra, let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ be its complexification. If $\mathfrak{g}$ is solvable, then $\mathfrak{g}^{\mathbb{C}}$ is solvable, so $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$ is nilpotent by (ii). Since $[\mathfrak{g}, \mathfrak{g}] \subset [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$, we conclude that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

The next theorem has a formulation similar to that of Lie's Theorem, but for nilpotent Lie groups or Lie algebras. However it should be remarked that the theorem holds for any field, even in positive characteristic.

Theorem 4.2. Engel's Theorem

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ *be a linear Lie algebra over a field* $\mathbb{K}$, V *a* $\mathbb{K}$*-vector space, and suppose that for any* $X \in \mathfrak{g}$, $X^n = 0$ *for some* $n \in \mathbb{N}$, *that is any element of* $\mathfrak{g}$ *is a nilpotent* *transformation. Then there exists a basis of V with respect to which $\mathfrak{g}$ is strictly upper triangular.* ♡

Remark It is not necessarily true that if $\mathfrak{g}$ is nilpotent it is upper triangular. In fact, the Lie algebra of diagonal matrices is nilpotent (since it is Abelian) but it does not necessarily have a realization in which it is strictly upper triangular.

However we have:

Corollary 4.6

A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(\mathfrak{g})$ is strictly upper triangular. ♡

Proof ($\Rightarrow$) If $\mathfrak{g}$ is nilpotent then $\mathrm{ad}(\mathfrak{g})^n = 0$, that is $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent for all $X \in \mathfrak{g}$. By Engel's Theorem $\mathrm{ad}(\mathfrak{g})$ is strictly upper triangular.

($\Leftarrow$) If $\mathrm{ad}(\mathfrak{g})$ is strictly upper triangular, then it is nilpotent, and from the short exact sequence $\{0\} \to Z(\mathfrak{g}) \to \mathfrak{g} \to \mathrm{ad}(\mathfrak{g}) \to \{0\}$ we conclude that $\mathfrak{g}$ is nilpotent as well. □

Definition 4.9

Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$. A vector $v \in V \setminus \{0\}$ is a common null vector of *$\{\rho(X) : X \in \mathfrak{g}\}$ if $\rho(X)v = 0$ for all $X \in \mathfrak{g}$.* ♣

To prove Engel's Theorem 4.2 it is enough to prove the following:

Theorem 4.3

Let $\mathfrak{g}$ be a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a representation such that $\rho(X)$ is nilpotent for every $X \in \mathfrak{g}$. Then $\rho(\mathfrak{g})$ has a common null vector in V. ♡

In fact, if this is true and V_0 is the space of common null vectors, then we can write $\rho(\mathfrak{g}) \subset \begin{pmatrix} 0_{\dim V_0} & * \\ 0 & * \end{pmatrix}$ and proceed inductively by considering $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V/V_0)$.

Lemma 4.4

Let $\mathfrak{g} \subseteq \mathfrak{gl}(fg)$ be a Lie algebra. If $X \in \mathfrak{g}$ is nilpotent, then $\mathrm{ad}(X)$ is nilpotent. ♡

Proof If $X \in \mathfrak{g}$, let $\mathfrak{l}_X, \mathfrak{r}_X \in \mathrm{End}(\mathfrak{g})$ be the commuting endomorphisms defined as

$$\mathfrak{l}_X(Y) := XY \text{ and } \mathfrak{r}_X(Y) := YX\,.$$

Since X is nilpotent, $\mathfrak{l}$ and $\mathfrak{r}$ are nilpotent as well. Thus $\mathrm{ad}(X) = \mathfrak{l}_X - \mathfrak{r}_X \in \mathrm{End}(\mathfrak{g})$ is nilpotent. □

Proof [Proof of Theorem 4.3] The proof will be by induction on $\dim \mathfrak{g}$ and in fact it will be similar to the proof of Lie's Theorem with the due modifications.

Let us assume that $\dim \mathfrak{g} = 1$ and let $0 \neq X \in \mathfrak{g}$. Since $\rho(X)$ nilpotent, let $n \in \mathbb{N}$ be the smallest integer such that $\rho(X)^n = 0$. Then there exists $v \in V$ such that $\rho(X)^{n-1}v \neq 0$. But $0 = \rho(X)^n v = \rho(X)(\rho(X)^{n-1}v)$, so $\rho(X)^{n-1}v$ is a null vector of X and thus of $\mathbb{R}X = \mathfrak{g}$.

Let us suppose now that $\dim \mathfrak{g} > 1$. Assume that every Lie algebra of dimension smaller than $\dim \mathfrak{g}$ satisfies the theorem. We can assume that ρ is faithful, otherwise $\mathfrak{g}/\ker\rho$ would be a Lie algebra of smaller dimension, for which the assertion is true. The proof consists of two steps:

1. We write $\mathfrak{g} = \mathbb{R}X_0 \oplus \mathfrak{h}$ where $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.
2. We find a null vector for X_0 in the space of common null vectors for $\mathfrak{h}$.

1. Let $\mathfrak{h}$ be a maximal proper subalgebra and consider $\mathrm{ad}_{\mathfrak{g}}(X)\colon \mathfrak{g} \to \mathfrak{g}$ for $X \in \mathfrak{g}$. If $X \in \mathfrak{h}$, since $\mathfrak{h}$ is a subalgebra we have $\mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{h}) \subset \mathfrak{h}$. Thus there is a representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$, $\mathrm{ad}\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$. Since $\mathfrak{g} \cong \rho(\mathfrak{g})$ consists of nilpotent elements, so does $\mathfrak{h}$, and hence so does $\mathrm{ad}(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ by Lemma 4.4. By the inductive hypothesis there exists $0 \neq X_0 \in \mathfrak{g}/\mathfrak{h}$ such that $\mathrm{ad}(\mathfrak{h})X_0 = 0 \in \mathfrak{g}/\mathfrak{h}$, that is $\mathrm{ad}(\mathfrak{h})X_0 \subset \mathfrak{h}$. Thus

$$[\mathbb{R}X_0 \oplus \mathfrak{h}, \mathbb{R}X_0 \oplus \mathfrak{h}] = \mathbb{R}[X_0, X_0] + \mathbb{R}[X_0, \mathfrak{h}] + \mathbb{R}[\mathfrak{h}, \mathfrak{h}] \subset 0 + \mathfrak{h} + \mathfrak{h} = \mathfrak{h},$$

that is $\mathbb{R}X_0 \oplus \mathfrak{h}$ is a subalgebra that contains $\mathfrak{h}$. Since $\mathfrak{h}$ was maximal, $\mathbb{R}X_0 \oplus \mathfrak{h} = \mathfrak{g}$ and $\mathfrak{h}$ is an ideal.

2. By inductive hypothesis the space W of null vectors of $\rho(\mathfrak{h})$ is non-zero. If we can prove that it is X_0-invariant, then we can apply the inductive hypothesis to $\rho\colon \mathbb{R}X_0 \to \mathfrak{gl}(W)$ and find a null vector in W that will be a null vector for $\mathfrak{g}$. So let $w \in W$, that is $\rho(X)w = 0$ for all $X \in \mathfrak{h}$. We want to show that $\rho(X_0)w \in W$, that is $\rho(X)\rho(X_0)w = 0$ for all $X \in \mathfrak{h}$. Since $\mathfrak{h}$ is an ideal, then $XX_0 - X_0X = [X, X_0] \in \mathfrak{h}$. Thus

$$\rho(X)\rho(X_0)w = \rho([X, X_0])w + \rho(X_0)\rho(X)w = 0 + \rho(X_0)0 = 0.$$

□

In the course of the proof, we have also obtained the following:

Corollary 4.7

If $\mathfrak{g}$ is nilpotent, and $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ is an ideal, $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h}$ and $\mathfrak{h}$ has codimension 1.

We have proved this under the assumption that elements of $\mathfrak{g}$ are nilpotent. The proof of this fact only needed that elements of $\mathrm{ad}(\mathfrak{g})$ are nilpotent, and so assuming that $\mathfrak{g}$ is nilpotent is enough.

Definition 4.10. Unipotent linear group

Let V be a vector space over a field $\mathbb{K}$ and let $G < \mathrm{GL}(V)$ be a linear group. Then G is unipotent *if*

$$G \leq \{g \in \mathrm{GL}(V) : (g - \mathrm{Id})^n = 0\},$$

where $n = \dim V$.

Corollary 4.8

If $G < \mathrm{GL}(V)$ is a unipotent group there exists a basis of V such that $G \leq \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}(V)\right\}$. Thus any unipotent connected group is nilpotent.

4.2.1 The Killing Form

There are other characterizations of nilpotent and solvable Lie algebras that use the Killing form, a particular case of the trace form.

Let V be a vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We recall that the trace of $A \in \mathrm{End}(V)$ is defined as $\mathrm{tr}A = \sum \lambda_i$, where λ_i are the eigenvalues of A. Note that a priori $\mathrm{tr}(A)$ will take values in the algebraic closure of $\mathbb{K}$. The following properties are satisfied:

1. $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}A$ for all $X \in \mathrm{GL}(V)$, that is $\mathrm{tr}\colon \mathrm{GL}(V) \to \mathbb{K}$ is independent of the choice of basis in V.
2. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$;
3. If we choose a basis of V and let $(a_{ij})_{ij}$ be the matrix representation of A with respect to it, then $\mathrm{tr}A = \sum a_{ii}$.

Definition 4.11. Trace form and Killing form

Let $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$.

1. *The* trace form *is the bilinear symmetric form*

$$\begin{aligned} B\colon \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} &\longrightarrow \quad \mathbb{K} \\ (X,Y) \qquad &\mapsto \operatorname{tr}(XY)\,. \end{aligned}$$

   Let $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{K}$.

2. *If* $\rho\colon \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{K})$ *is a Lie algebra representation, the* trace form of ρ *is*

$$\begin{aligned} B_\rho\colon \mathfrak{g}\times\mathfrak{g} &\longrightarrow \qquad\qquad \mathbb{K} \\ (X,Y) &\mapsto B(\rho(X),\rho(Y)) = \operatorname{tr}(\rho(X)\rho(Y))\,. \end{aligned}$$

3. *The* Killing form *of* $\mathfrak{g}$ *is* $B_{\mathfrak{g}} := B_{\mathrm{ad}}\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$*, that is*

$$B_{\mathfrak{g}}(X,Y) = \operatorname{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))\,.$$

♣

Definition 4.12. Invariant bilinear form

Let V *be a vector space over* $\mathbb{K}$*,* $f\colon V\times V \to \mathbb{K}$ *a bilinear form,* $G \le \mathrm{GL}(V)$*,* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$*.*

1. f *is* G-invariant *if* $f(AX,AY) = f(X,Y)$ *for all* $X,Y\in V$ *and all* $A\in G$*.*
2. f *is* $\mathfrak{g}$*-invariant if* $f(DX,Y) + f(X,DY) = 0$ *for all* $X,Y\in V$ *and all* $D\in\mathfrak{g}$*.*

♣

Proposition 4.7

Let V *be a finite dimensional vector space over* $\mathbb{K}$ $f\colon V\times V\to\mathbb{K}$ *be a bilinear form and* $A\in End(V)$*. The following are equivalent:*

1. $f(AX,Y) + f(X,AY) = 0$ *for all* $X,Y\in V$
2. $f((\exp tA)X, (\exp tA)Y) = f(X,Y)$ *for all* $X,Y\in V$ *and all* $t\in\mathbb{R}$*.*

♠

Proof (2. $\Leftarrow$ 1.) This follows just from differentiating the expression $f((\exp tA)X,(\exp tA)Y) = f(X,Y)$.

(1. $\Rightarrow$ 2.) We will show that $\phi(t) := f((\exp tA)X,(\exp tA)Y)$ and $\psi(t) = f(X,Y)$ are both solutions of the differential equation $\frac{dz}{dt} = 0$ with $z(0) = f(X,Y)$. Obviously this is the case for $\psi(t)$. By differentiating $\phi(t)$ we obtain

$$\phi'(t) = f(A(\exp tA)X,(\exp tA)Y) + f((\exp tA)X, A(\exp tA)Y)\,.$$

Then using 1. after replacing X by $(\exp tA)X$ and Y by $(\exp tA)Y$, we obtain that $\phi'(t) = 0$. □

Corollary 4.9

Let $G \leq \mathrm{GL}(V)$ be a closed subgroup with Lie algebra $\mathfrak{g}$ and let $f : V \times V \to \mathbb{R}$ be a bilinear form. Then f is G-invariant if and only if it is $\mathfrak{g}$-invariant.

Proposition 4.8

Let $\mathfrak{g}$ be a Lie algebra, $\rho\colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{K})$ a representation. Then B_ρ is $\mathrm{ad}(\mathfrak{g})$-invariant. In particular the Killing form is $\mathrm{ad}(\mathfrak{g})$-invariant.

Proof Recall that if $S, T \in \mathrm{End}(V)$, then $\mathrm{tr}(ST) = \mathrm{tr}(TS)$. If $X, Y, Z \in \mathfrak{g}$ then

$$\begin{aligned} B_\rho(\mathrm{ad}(X)Y, Z) &= B_\rho([X, Y], Z) = \mathrm{tr}(\rho([X, Y])\rho(Z)) \\ &= \mathrm{tr}(\rho(X)\rho(Y)\rho(Z) - \rho(Y)\rho(X)\rho(Z)) \\ &= \mathrm{tr}(\rho(Y)\rho(Z)\rho(X) - \rho(Y)\rho(X)\rho(Z)) \\ &= \mathrm{tr}(\rho(Y)\rho([Z, X]) = -\mathrm{tr}(\rho(Y)\rho([X, Z])) \\ &= -B_\rho(Y, [X, Z]) = -B_\rho(Y, \mathrm{ad}(X)Z). \end{aligned}$$

□

We will see that the Killing form is a powerful tool in the theory of Lie groups and Lie algebras. For example we have:

Corollary 4.10. Cartan's Criterion for solvability

Let $\mathfrak{g}$ be a Lie algebra with Killing form $B_\mathfrak{g}$. Then $\mathfrak{g}$ is solvable if and only if $B_\mathfrak{g}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} \equiv 0$.

The proof of Cartan's Criterion relies upon the following theorem, where the meat of the argument is and which we prove at the end of this section.

Theorem 4.4

Let V be a finite dimensional complex vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra. If $\mathrm{tr}(XY) = 0$ for all $X, Y \in \mathfrak{g}$, then there exists a basis of V with respect to which $\mathfrak{g}^{(1)}$ is strictly upper triangular. In particular $\mathfrak{g}^{(1)}$ is nilpotent and $\mathfrak{g}$ is solvable.

We will start the proof of Cartan's Criterion with a preliminary result:

Lemma 4.5

Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Then $B_\mathfrak{h} = B_\mathfrak{g}|_{\mathfrak{h} \times \mathfrak{h}}$.

Remark To have that $\mathrm{ad}_\mathfrak{h}(X) = \mathrm{ad}_\mathfrak{g}(X)|_\mathfrak{h}$ for all $X \in \mathfrak{h}$ it is enough that $\mathfrak{h}$ is a subalgebra. To say that $B_\mathfrak{h} = B_\mathfrak{g}|_{\mathfrak{h} \times \mathfrak{h}}$ we need that $\mathfrak{h}$ is an ideal.

Proof [Proof of Lemma 4.5] Let V be a linear complement of $\mathfrak{h}$, so $\mathfrak{g} = \mathfrak{h} \oplus V$. Then $\mathrm{ad}_{\mathfrak{g}}(X) : \mathfrak{h} \oplus V \to \mathfrak{h} \oplus V$ is such that if $X \in \mathfrak{h}$, then

1. $\mathrm{ad}_{\mathfrak{g}}(X)Y = [X, Y] \in \mathfrak{h}$ if $Y \in \mathfrak{h}$ (since $\mathfrak{h}$ is a subalgebra) but also
2. $\mathrm{ad}_{\mathfrak{g}}(X)Y = [X, Y] \in \mathfrak{h}$ if $Y \in V$ (since $\mathfrak{h}$ is an ideal).

Hence

$$\mathrm{ad}_{\mathfrak{g}}(X) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{h}}(X) & * \\ \underbrace{0}_{\mathfrak{h}\text{ is a subalgebra}} & \underbrace{0}_{\mathfrak{h}\text{ is an ideal}} \end{pmatrix}$$

so that $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(X)\mathrm{ad}_{\mathfrak{g}}(Y)) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{h}}(X)\mathrm{ad}_{\mathfrak{h}}(Y))$ for all $X, Y \in \mathfrak{h}$. □

Proof [Proof of Corollary 4.10] ($\Rightarrow$) Suppose that $\mathfrak{g}$ is solvable. Then by Proposition 4.6 $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Hence by Corollary 4.6 $\mathrm{ad}(\mathfrak{g}^{(1)})$ is strictly upper triangular. This implies that $B_{\mathfrak{g}}|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}} = B_{\mathfrak{g}^{(1)}} = 0$, where we applied the previous lemma to the ideal $\mathfrak{g}^{(1)} \lhd \mathfrak{g}$.

($\Leftarrow$) By Theorem 4.4 applied to $\mathrm{ad}(\mathfrak{g})^{(1)} = \mathrm{ad}(\mathfrak{g}^{(1)})$, if $X, Y \in \mathfrak{g}^{(1)}$ and $0 = B_{\mathfrak{g}}(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$, then $[\mathrm{ad}(\mathfrak{g}^{(1)}), \mathrm{ad}(\mathfrak{g}^{(1)})] = \mathrm{ad}([\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]) = \mathrm{ad}(\mathfrak{g}^{(2)})$ is strictly upper triangular and hence nilpotent, hence solvable. We need to show that $\mathfrak{g}$ is solvable.

Since $\mathrm{ad}(\mathfrak{g}^{(2)})$ is a solvable ideal in $\mathrm{ad}(\mathfrak{g}^{(1)})$ and $\mathrm{ad}(\mathfrak{g}^{(1)})/\mathrm{ad}(\mathfrak{g}^{(2)}) = \mathrm{ad}(\mathfrak{g}^{(1)}/\mathfrak{g}^{(2)})$ is Abelian, hence solvable, $\mathrm{ad}(\mathfrak{g}^{(1)})$ is solvable. Analogously, $\mathrm{ad}(\mathfrak{g}^{(1)})$ is a solvable ideal in $\mathrm{ad}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})/\mathrm{ad}(\mathfrak{g}^{(1)}) = \mathrm{ad}(\mathfrak{g}/\mathfrak{g}^{(1)})$ is Abelian, hence solvable, so $\mathrm{ad}(\mathfrak{g})$ is solvable. Finally the short exact sequence $0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to \mathrm{ad}(\mathfrak{g}) \to 0$ shows that $\mathfrak{g}$ is solvable. □

In order to prove Theorem 4.4 we need a result that is a corollary of the Jordan canonical form over $\mathbb{C}$.

Proposition 4.9

Let V be a finite dimensional vector space over $\mathbb{C}$ and let $A \in \mathrm{End}(V)$. Then there exist a diagonalizable $S \in \mathrm{End}(V)$ and a nilpotent $N \in \mathrm{End}(V)$ such that

1. *$A = S + N$;*
2. *$SN = NS$;*
3. *S and N are uniquely determined, and*
4. *there exist polynomials $s(X), n(X) \in \mathbb{C}[X]$ without constant terms, such that*

$$S = s(A) \quad \text{and} \quad N = n(A)\,.$$

♠

We say that N is the *nilpotent component* of A amd S is the *semisimple component* of A. For a proof of Proposition 4.9 see [8, Proposition 12.19].

Example 4.6 Let $X \in \mathfrak{gl}(n,\mathbb{C})$. Then $X = S+N$ and $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{gl}(n,\mathbb{C}))$ has a decomposition

$$\mathrm{ad}(X) = \mathrm{ad}(S) + \mathrm{ad}(N)\,. \tag{4.5}$$

In fact:

1. S is diagonalizable and hence $\mathrm{ad}(S)$ is diagonalizable;
2. N is nilpotent and hence $\mathrm{ad}(N)$ is nilpotent;
3. $[N,S] = 0$ and hence $[\mathrm{ad}(N), \mathrm{ad}(S)] = \mathrm{ad}([N,S]) = 0$.

Thus (4.5) is the decomposition of $\mathrm{ad}(X)$. Moreover $\mathrm{ad}(N)$ and $\mathrm{ad}(S)$ are polynomials in $\mathrm{ad}(X)$. Notice that this is not obvious a priori, since ad is only a Lie algebra homomorphism and not an algebra homomorphism. Hence if $S = s(X)$ and $N = n(X)$, there exist $n', s' \in \mathbb{C}[X]$ such that

$$n'(\mathrm{ad}(X)) = \mathrm{ad}(N) \quad \text{and} \quad s'(\mathrm{ad}(X)) = \mathrm{ad}(S)\,. \tag{4.6}$$

In particular if $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$ and $X \in \mathfrak{g}$, it follows from (4.6) that $\mathrm{ad}(S)$ and $\mathrm{ad}(N)$ leave $\mathfrak{g}$ invariant.

Proof [Proof of Theorem 4.4] Because of Engel's Theorem, it will be enough to prove that every $A \in \mathfrak{g}^{(1)}$ is nilpotent. We know from Proposition 4.9 that $A = S + N$, where S is diagonalizable and N is nilpotent. Hence it will be enough to show that $S = 0$. To this purpose we will show that if

$$S = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{and} \quad \overline{S} = \begin{pmatrix} \overline{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda}_n \end{pmatrix},$$

then $\mathrm{tr}(S\overline{S}) = \sum \lambda_i \overline{\lambda}_j = 0$, hence showing that $S = 0$.

Since $A = S + N$, we have that

$$\mathrm{tr}(S\overline{S}) = \mathrm{tr}((A-N)\overline{S}) = \mathrm{tr}(A\overline{S}) - \mathrm{tr}(N\overline{S}),.$$

Observe that, by hypothesis, $\mathrm{tr}(XY) = 0$ for all $X, Y \in \mathfrak{g}$, but a priori S and N are only in $\mathfrak{gl}(V)$ and not necessarily in $\mathfrak{g}$. We will hence show the following facts:

1. $\mathrm{tr}(A\overline{S}) = 0$;
2. $\mathrm{tr}(N\overline{S}) = 0$.

To prove both statements it will be useful to consider a polynomial $p \in \mathbb{C}[X]$ such that $p(\lambda_i) = \overline{\lambda}_i$ for $i = 1, \dots, n$. For example one can take $p(x) = \sum_{j=1}^n \overline{\lambda}_j \prod_{i \neq j} \frac{x-\lambda_i}{\lambda_j - \lambda_i}$. Then

$p(S) = \overline{S}$.

To prove 1. remember that $A \in \mathfrak{g}^{(1)}$, so that $A = \sum_{i=1}^{k} [X_i, Y_i]$ for $x_i, Y_i \in \mathfrak{g}$. Then

$$\begin{aligned} \operatorname{tr}(A\overline{S}) &= \operatorname{tr}\left(\sum_{i=1}^{k} [X_i, Y_i]\overline{S}\right) = \operatorname{tr}\left(\sum_{i=1}^{k} (X_iY_i)\overline{S} - \sum_{i=1}^{k} (Y_iX_i)\overline{S}\right) \\ &= \sum_{i=1}^{k} \operatorname{tr}(Y_i\overline{S}X_i - \overline{S}Y_iX_i) = \sum_{i=1}^{k} \operatorname{tr}([Y_i, \overline{S}]X_i) = -\sum_{i=1}^{k} \operatorname{tr}((\operatorname{ad}(\overline{S})Y_i)X_i)\,. \end{aligned}$$

Since $X_i \in \mathfrak{g}$ and we know that $\operatorname{tr}(XY) = 0$ for all $X, Y \in \mathfrak{g}$, it will be enough to show that $\operatorname{ad}(\overline{S})Y_i \in \mathfrak{g}$, that is that $\operatorname{ad}(\overline{S})(\mathfrak{g}) \subseteq \mathfrak{g}$. But $\overline{S} = p(S)$ and S is a polynomial in A, so that $\overline{S}$ as well is a polynomial in A. By the Example 4.6, $\operatorname{ad}(q(A)) = q'(\operatorname{ad}(A))$, that is $\operatorname{ad}(q(A))\mathfrak{g} \subseteq \mathfrak{g}$.

To prove 2. recall that $[S, N] = 0$. Moreover $[\overline{S}, N] = 0$, since, if N commutes with S, it commutes with all powers of S, and hence with a polynomial in S. Thus $(N\overline{S})^\ell = N^\ell \overline{S}^\ell = 0$ as soon as $N^\ell = 0$. □

4.3 Semisimplicity

We start by describing Lie algebras that, contrary to nilpotent and solvable ones, have no non-trivial ideals.

Definition 4.13. (Semi)simplicity

a) A Lie algebra $\mathfrak{g}$ is simple *if*

(i) It is not Abelian;

(ii) Its only ideals are $\{0\}$ and $\mathfrak{g}$.

b) A Lie algebra is semisimple *if it is the direct sum of simple ideals.*

c) A connected Lie group is simple *(respectively* semisimple*) if its Lie algebra is simple (respectively semisimple).* ♣

Theorem 4.5. (Dieudonné)

Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if $B_{\mathfrak{g}}$ is non-degenerate. ♡

Remark 4.3 If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus V$, then using that $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{h})(\mathfrak{g}) \subset \mathfrak{h}$ and $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{g})(\mathfrak{h}) \subset \mathfrak{h}$, we have that

$$\operatorname{ad}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \begin{pmatrix} \operatorname{ad}_{\mathfrak{h}}(\mathfrak{h}) & * \\ 0 & 0 \end{pmatrix} \text{ and } \operatorname{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Hence if $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, then

$$\mathrm{ad}_{\mathfrak{g}}(X)\mathrm{ad}_{\mathfrak{g}}(Y) \subseteq \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$

We will need the following lemma.

Lemma 4.6

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Then

$$\mathfrak{h}^{\perp} = \{X \in \mathfrak{g} : B_{\mathfrak{g}}(X, A) = 0 \textit{ for all } A \in \mathfrak{h}\}$$

is also an ideal. ♡

Proof Let $X \in \mathfrak{h}^{\perp}$, that is $B_{\mathfrak{g}}(X, A) = 0$ for all $A \in \mathfrak{h}$. We want to show that for all $Y \in \mathfrak{g}$ also $B_{\mathfrak{g}}([X, Y], A) = 0$ for all $A \in \mathfrak{h}$. In fact:

$$B_{\mathfrak{g}}([X, Y], A) = -B_{\mathfrak{g}}(\mathrm{ad}(Y)(X), A) = B_{\mathfrak{g}}(X, \mathrm{ad}(Y)(A)) = B_{\mathfrak{g}}(X, [Y, A]) = 0$$

since $[Y, A] \in \mathfrak{h}$. □

Proof [Proof of Theorem 4.5] ($\Rightarrow$) Since $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where the $\mathfrak{g}_i$ are simple ideals, then $B_{\mathfrak{g}} = \sum B_{\mathfrak{g}_i}$ and we may as well assume that $\mathfrak{g}$ is simple. Let $\mathfrak{g}^{\perp} = \{Y \in \mathfrak{g} : B_{\mathfrak{g}}(X, Y) = 0$ for all $X \in \mathfrak{g}\}$. Then $\mathfrak{g}^{\perp}$ is an ideal by the previous lemma, and since $\mathfrak{g}$ is simple either $\mathfrak{g}^{\perp} = (0)$ or $\mathfrak{g}^{\perp} = \mathfrak{g}$. If $\mathfrak{g}^{\perp} = \mathfrak{g}$, then $B_{\mathfrak{g}} \equiv 0$, and so $\mathfrak{g}$ is solvable by Cartan's criterion. Hence $\mathfrak{g}^{\perp} = (0)$, and so $B_{\mathfrak{g}}$ is non-degenerate.

($\Leftarrow$) The proof will follow the following steps. Assume that $B_{\mathfrak{g}}$ is non-degenerate.

1. There are no Abelian ideals.
2. If $\mathfrak{h} \subset \mathfrak{g}$ is a non-trivial ideal, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$.
3. $B_{\mathfrak{h}}$ and $B_{\mathfrak{h}^{\perp}}$ are non-degenerate.
4. Argue by induction.

1. If $\mathfrak{a}$ were an Abelian ideal, then $Z_{\mathfrak{a}}(\mathfrak{a}) = \mathfrak{a}$ and so $\mathrm{ad}_{\mathfrak{g}}(A) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ for all $A \in \mathfrak{a}$. Thus for all $Y \in \mathfrak{g}$

$$B_{\mathfrak{g}}(A, Y) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(A)\mathrm{ad}_{\mathfrak{g}}(Y)) = \mathrm{tr}\left(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right) = \mathrm{tr}\left(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right) = 0\,,$$

hence $B_{\mathfrak{g}}$ would be degenerate.

2. Because of the previous lemma we need to check that $\mathfrak{h} \cap \mathfrak{h}^{\perp} = (0)$. In fact:

(a). $\mathfrak{h} \cap \mathfrak{h}^\perp$ is an ideal, being the intersection of ideals.

(b). $\mathfrak{h} \cap \mathfrak{h}^\perp$ is Abelian. In fact if $X, Y \in \mathfrak{h} \cap \mathfrak{h}^\perp$, then for all $Z \in \mathfrak{g}$:

$$B_{\mathfrak{g}}([X,Y], Z) = -B_{\mathfrak{g}}(Y, [X,Z]) \in B_{\mathfrak{g}}(\mathfrak{h}, \mathfrak{h}^\perp) = 0\,.$$

Since $B_{\mathfrak{g}}$ is non-degenerate, this implies that $[X,Y] = 0$ for all $X, Y \in \mathfrak{h} \cap \mathfrak{h}^\perp$.

Thus (a) and (b) imply that $\mathfrak{h} \oplus \mathfrak{h}^\perp \subseteq \mathfrak{g}$. Moreover, since $B_{\mathfrak{g}}$ is non-degenerate, $\dim \mathfrak{g} = \dim \mathfrak{h} + \dim \mathfrak{h}^\perp - \dim \mathfrak{h} \cap \mathfrak{h}^\perp = \dim \mathfrak{h} + \dim \mathfrak{h}^\perp$, and so equality holds.

3. Let $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ be such that $B_{\mathfrak{g}}(X,Y) \neq 0$ and let $Y = Y_{\mathfrak{h}} + Y_{\mathfrak{h}^\perp}$ with $Y_{\mathfrak{h}} \in \mathfrak{h}$ and $Y_{\mathfrak{h}^\perp} \in \mathfrak{h}^\perp$. Then

$$\begin{aligned}
0 \neq B_{\mathfrak{g}}(X,Y) &= \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)\operatorname{ad}_{\mathfrak{g}}(Y)) = \operatorname{tr}\left(\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right) = \operatorname{tr}\left(\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\right) = \\
&= \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)\operatorname{ad}_{\mathfrak{g}}(Y)|_{\mathfrak{h}}) = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)\operatorname{ad}_{\mathfrak{g}}(Y_{\mathfrak{h}})|_{\mathfrak{h}} + \operatorname{ad}_{\mathfrak{g}}(X)\operatorname{ad}_{\mathfrak{g}}(Y_{\mathfrak{h}^\perp}))x3|_{\mathfrak{h}} = \\
&\overset{(*)}{=} \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)\operatorname{ad}_{\mathfrak{g}}(Y_{\mathfrak{h}})|_{\mathfrak{h}}) \overset{(**)}{=} \operatorname{tr}(\operatorname{ad}_{\mathfrak{h}}(X)\operatorname{ad}_{\mathfrak{h}}(Y_{\mathfrak{h}})) = B_{\mathfrak{h}}(X, Y_{\mathfrak{h}})\,,
\end{aligned}$$

where $(*)$ follows since $\operatorname{ad}(\mathfrak{h}^\perp)|_{\mathfrak{h}} = 0$ because $[\mathfrak{h}^\perp, \mathfrak{h}] \subset \mathfrak{h}^\perp \cap \mathfrak{h} = \{0\}$ and $(**)$ from Remark 4.3. Thus $B_{\mathfrak{h}}$ is non-degenerate. Similarly, $\mathfrak{h}^\perp$ is non-degenerate.

4. If $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are simple we have found a decomposition of $\mathfrak{g}$ as direct sum of simple ideals. If either one is not, we choose a non-trivial ideal and repeat the argument. Since $\dim \mathfrak{h} < \dim \mathfrak{g}$, the process ends with a decomposition as direct sum of simple ideals. □

To see how to determine the non-degeneracy of the Killing form, we prove the following theorem:

Theorem 4.6

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ is self-adjoint with respect to some inner product in $\mathbb{K}^n$ and $Z_{\mathfrak{g}}(\mathfrak{g}) = \{0\}$, then $B_{\mathfrak{g}}$ is non-degenerate. ♡

Remark 1. Since $\ker \operatorname{ad} = Z(\mathfrak{g})$ is an ideal, then $Z(\mathfrak{g}) = \{0\}$ is obviously necessary.

2. There is a theorem of Mostow that states that any semisimple Lie algebra has a faithful linear representation whose image is self-adjoint, hence the above theorem gives somehow a necessary and sufficient condition for semisimplicity.

Define $Z_{\mathfrak{g}}(\mathfrak{g})$ and decide about the subscript

Example 4.7

1. $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{sl}(n, \mathbb{R})^*$ and $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C})^*$.

2. $\mathfrak{so}(n, \mathbb{R}) = \mathfrak{so}(n, \mathbb{R})^*$, $n \geq 3$.
3. $\mathfrak{so}(p, q) = \mathfrak{so}(p, q)^*$.
4. $\mathfrak{su}(n) = \mathfrak{su}(n)^*$.

Hence these are all semisimple (and in fact, simple) Lie algebras.

Lemma 4.7

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1. *The trace form is non-degenerate.*
2. *If $W \subset \mathbb{K}^{n\times n}$ is a subspace that is self-adjoint with respect to some inner product in $\mathbb{K}^n$, then $B|_{W\times W}$ is non-degenerate.*

♡

Proof 1. For $X \in \mathbb{K}^{n\times n}$, let X^* be the adjoint with respect to the usual inner product in $\mathbb{K}^n$, that is $X^* = {}^tX$ if $\mathbb{K} = \mathbb{R}$ and $X^* = \overline{{}^tX}$ if $\mathbb{K} = \mathbb{C}$. Let us define

$$B^*(X, Y) := B(X, Y^*) = \mathrm{tr}(XY^*) = \sum_{i,j} X_{ij}\overline{Y}_{ij}$$

Then B^* is the usual inner product on $\mathbb{K}^{n\times n}$ and $B(X, X) = B^*(X, X^*) = \|X\|^2 > 0$ if $X \neq 0$, so that B is non-degenerate.

2. Follows from the proof of 1., since if $X \in W$ then $X^* \in W$. □

Proof [Proof of Theorem 4.6] We start with a few observations. Let $\mathrm{tr}\colon \mathfrak{gl}(n, \mathbb{K}) \times \mathfrak{gl}(n, \mathbb{K}) \to \mathbb{K}$ be the trace form. Then the Killing form is the composition of the following maps:

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\mathrm{ad}_{\mathfrak{g}} \times \mathrm{ad}_{\mathfrak{g}}} \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \times \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \xrightarrow{i} \mathfrak{gl}(\mathfrak{g}) \times \mathfrak{gl}(\mathfrak{g}) \xrightarrow{\mathrm{tr}} \mathbb{K}.$$

If $Z(\mathfrak{g}) = \{0\}$ then $\mathfrak{g} \cong \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$, so it will be enough to show that:

Claim 4.3.1. *If $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$ is self-adjoint with respect to an inner product in $\mathbb{K}^n$, then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$ is self-adjoint with respect to an inner product in $\mathbb{K}^{n\times n}$.*

In fact, assuming the claim, we can apply Lemma 4.7 (2) with $W = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ and conclude the proof.

To prove the claim we will need two lemmas.

Lemma 4.8

Let $\langle\cdot,\cdot\rangle_+\colon \mathfrak{gl}(n, \mathbb{K}) \times \mathfrak{gl}(n, \mathbb{K}) \to \mathbb{K}$ be the inner product defined by $\langle X, Y\rangle_+ = \mathrm{tr}(XY^*)$, and let $\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}\colon \mathfrak{gl}(n, \mathbb{K}) \to \mathfrak{gl}(\mathfrak{gl}(n, \mathbb{K}))$ be the adjoint representation of $\mathfrak{gl}(n, \mathbb{K})$.

Then $\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)^* = \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A^*)$*, where* A^* *is the adjoint with respect to the usual inner product in* $\mathbb{K}^{n\times n}$ *and* $\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)^*$ *is the adjoint with respect to the inner product* $\langle\cdot,\cdot\rangle_+$*.* ♡

Proof We need to verify that $\langle X, \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A^*)Y\rangle_+ = \langle \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)X, Y\rangle_+$. In fact

$$\begin{aligned}\langle X, \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A^*)Y\rangle_+ = \langle X, [A^*, Y]\rangle_+ &= \mathrm{tr}(X[A^*,Y]^*) = \mathrm{tr}(X(A^*Y - YA^*)^*) \\ &= \mathrm{tr}(X(Y^*A - AY^*)) = \mathrm{tr}(XY^*A) - \mathrm{tr}(XAY^*) \\ &= \mathrm{tr}(AXY^*) - \mathrm{tr}(XAY^*) = \mathrm{tr}((AX - XA)Y^*) \\ &= \mathrm{tr}([A,X]Y^*) = \langle [A,X], Y\rangle_+ = \langle \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)X, Y\rangle_+.\end{aligned}$$

□

The point of the lemma is that now we know that if $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{K})$ is self-adjoint, then $\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{gl}(n,\mathbb{K}))$ is self-adjoint. This is however not quite enough, as we want to see that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$ is self-adjoint. Since $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{K})$ is a subalgebra, it is easy to see that

$$\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(\mathfrak{g}) \subseteq \begin{pmatrix} \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) & * \\ 0 & 0 \end{pmatrix},$$

that is if $A \in \mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{K})$, then $\mathrm{ad}_{\mathfrak{g}}(A) = \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)|_{\mathfrak{g}}$. It will be hence enough to prove the following lemma, with $\mathfrak{g} = V$, $\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(\mathfrak{g}) = \mathfrak{h}$ and $n^2 = m$.

Lemma 4.9

Let $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*. If* $\mathfrak{h} \subset \mathfrak{gl}(m,\mathbb{K})$ *is a self-adjoint Lie algebra, with respect to some inner product on* $\mathbb{K}^m$ *and* $V \subset \mathbb{K}^m$ *is an* $\mathfrak{h}$*-invariant subspace, then* $\mathfrak{h}|_V$ *is self-adjoint.* ♡

Proof Since $\mathfrak{h} = \mathfrak{h}^*$, then if V is $\mathfrak{h}$-invariant so is $V^\perp$. In fact, let $\mathbb{K}^m = V \oplus V^\perp$. For all $H \in \mathfrak{h}$ we have $HV \subset V$, and if $v \in V^\perp$ we want to see that $Hv \in V^\perp$ as well for all $H \in \mathfrak{h}$. But $Hv \in V^\perp$ if and only if $\langle Hv, w\rangle = 0$ for all $w \in V$. In fact $\langle Hv, w\rangle = \langle v, H^*w\rangle = 0$, since $H^* \in \mathfrak{h}$ and so $H^*w \in V$. So if $H \in \mathfrak{h}$, we can write $H = H_V \oplus H_{V^\perp}$ and $H^* = H_V^* \oplus H_{V^\perp}^*$. □

□

Corollary 4.11

Let V *be a* $\mathbb{C}$*-vector space. If* $\mathfrak{g} \subset \mathfrak{sl}(V)$ *is an irreducible and self-adjoint Lie algebra (with respect to some inner product on* V*), then* $Z_{\mathfrak{g}}(\mathfrak{g}) = \{0\}$ *and hence* $B_{\mathfrak{g}}$ *is non-degenerate.* ♡

Here *irreducible* means as an algebra of endomorphisms that is, there are no non-trivial $\mathfrak{g}$-invariant subspaces in V. For example the Lie algebras $\mathfrak{sl}(V)$ and $\mathfrak{su}(n)$ act respectively on V and

$\mathbb{C}^n$ irreducibly, hence we deduce that they are semisimple.

The proof of Corollary 4.11 relies upon this classical result in representation theory. There are several version of it, but we will need the most elementary one.

Lemma 4.10. (Schur)

Let $\mathfrak{g}$ be a Lie algebra acting irreducibly on a complex vector space V and let $A\colon V \to V$ be an endomorphism that commutes with $\mathfrak{g}$. Then $A = cI$ for some $c \in \mathbb{C}$. ♡

Proof Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and consider the endomorphism $A - \lambda\mathrm{Id}$, which also commutes with $\mathfrak{g}$. It is straightforward to check that $\ker(A - \lambda\mathrm{Id})$ is a $\mathfrak{g}$-invariant subspace. Since the action of $\mathfrak{g}$ is irreducible and $\ker(A - \lambda\mathrm{Id}) \neq \{0\}$, then $\ker(A - \lambda\mathrm{Id}) = V$, that is $A = \lambda\mathrm{Id}$. □

Proof [Proof of Corollary 4.11] Let $A \in \mathfrak{g} \subset \mathfrak{gl}(V)$ be an endomorphism that commutes with $\mathfrak{g}$. Then $A = cI$ by Schur's Lemma. Now if $A \in Z_{\mathfrak{g}}(\mathfrak{g}) \subset \mathfrak{g} \subset \mathfrak{sl}(V)$, then $0 = \mathrm{tr} A = c \dim V$, and so $A = 0$. □

Remark The non-degeneracy of $B_{\mathfrak{g}}$ characterizes semisimple Lie algebras, but in general $B_{\mathfrak{g}}$ is not definite. In fact, we will see that the definiteness of $B_{\mathfrak{g}}$ is equivalent to compactness within the semisimple Lie algebras.

Before we move to the next section, we give a bit more details on the structure of semisimple Lie algebras.

Proposition 4.10

Let $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$ be the direct sum of simple ideals over some index set I. Then any ideal $\mathfrak{h} \subset \mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{i\in J} \mathfrak{g}_i$ with $J \subset I$. ♠

Proof Let $J \subset I$ be the smallest subset such that $\mathfrak{h} \subseteq \bigoplus_{i\in J} \mathfrak{g}_i$. We are going to show that there is equality. Let $i \in J$. Then $[\mathfrak{h}, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$, since $\mathfrak{g}_i$ is an ideal; moreover, since $[\mathfrak{h}, \mathfrak{g}_i]\mathfrak{g}_i$ is an ideal, either $[\mathfrak{h}, \mathfrak{g}_i] = \mathfrak{g}_i$ or $[\mathfrak{h}, \mathfrak{g}_i] = \{0\}$. We will show that $[\mathfrak{h}, \mathfrak{g}_i] \neq \{0\}$ for every $i \in J$, so that $\mathfrak{g}_i = [\mathfrak{h}, \mathfrak{g}_i] \subset \mathfrak{h}$, which implies that $\mathfrak{h} = \bigoplus_{i\in J} \mathfrak{g}_i$.

To see that $[\mathfrak{h}, \mathfrak{g}_i] \neq \{0\}$, let $X \in \mathfrak{h}$ be such that $X = X_1 + \cdots + X_n$ with $X_j \in \mathfrak{g}_j$ and $|J| = n$. If $[\mathfrak{h}, \mathfrak{g}_j] = \{0\}$, the in particular $[X, \mathfrak{g}_j] = \{0\}$, so that $[X_j, \mathfrak{g}_j] = \{0\}$. Thus $X_j \in Z_{\mathfrak{g}_j}(\mathfrak{g}_j) = 0$, contradicting the minimality of J. □

Corollary 4.12

1. *Any semisimple Lie algebra has a finite number of ideals.*
2. *Any connected semisimple Lie group with finite center has a finite number of connected normal subgroups.*

♡

Proposition 4.11

Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:

1. *$\mathfrak{g}$ is semisimple;*
2. *$\mathfrak{g}$ has no non-trivial Abelian ideals;*
3. *$\mathfrak{g}$ has no non-trivial solvable ideals.*

♠

Corollary 4.13

1. *G is a connected simple Lie group if and only if every connected normal proper subgroup of G is trivial. In particular the center $Z(G)$ of a connected simple Lie group is discrete.*
2. *G is a connected semisimple Lie group if and only if it has no non-trivial connected normal Abelian subgroups.*
3. *G is a connected semisimple Lie group if and only if it has no non-trivial connected normal solvable subgroups.*

♡

Remark To go from Proposition 4.11 to Corollary 4.13 one uses Proposition 3.16. Also, concerning 1. in Corollary 4.13, observe that, as we proved in Proposition 2.1.6., as G is connected, any discrete normal subgroup is contained in the center of G.

Proof [Proof of Proposition 4.11] $(1. \Rightarrow 2.)$ This is clear from Proposition 4.10.

$(2. \Rightarrow 3.)$ If there were to exist a solvable ideal $\mathfrak{h}$, then there would be a descending series of ideals $\mathfrak{h} \supset \mathfrak{h}^{(1)} \supset \cdots \supset \mathfrak{h}^{(n)} = \{0\}$ and $\mathfrak{h}^{(n-1)}$ would be Abelian. Moreover, since the $\mathfrak{h}^{(i)}$ are characteristic ideals in $\mathfrak{h}^{(i-1)}$, $\mathfrak{h}^{(n-1)}$ would be an Abelian ideal in $\mathfrak{g}$.

$(3. \Rightarrow 1.)$ It is enough to see that $B_{\mathfrak{g}}$ is non-degenerate. Let $\mathfrak{h} \subset \mathfrak{g}$ be the kernel of $B_{\mathfrak{g}}$, that is $\mathfrak{h} = \{X : B_{\mathfrak{g}}(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$. Since $B_{\mathfrak{g}}$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant, then $\mathfrak{h}$ is an ideal and, by Lemma 4.5, $B_{\mathfrak{h}} = B_{\mathfrak{g}}|_{\mathfrak{h}\times\mathfrak{h}}$. Thus $\mathfrak{h}$ would be solvable, which contradicts the hypothesis. Thus $\mathfrak{h} = \{0\}$ and hence $B_{\mathfrak{g}}$ is non-degenerate. $\square$

Proposition 4.12

If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Proof Let $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$, where $|I| < \infty$ and the $\mathfrak{g}_i$ are simple ideals. If $i \neq j$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_i \cap \mathfrak{g}_j = \{0\}$, while $[\mathfrak{g}_i, \mathfrak{g}_i]$ is an ideal in $\mathfrak{g}_i$. Since the $\mathfrak{g}_i$ are simple, hence in particular not Abelian, $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$. Thus

$$[\mathfrak{g}, \mathfrak{g}] = \left[\bigoplus_{i\in I} \mathfrak{g}_i, \bigoplus_{i\in I} \mathfrak{g}_i\right] = \bigoplus_{i\in I} [\mathfrak{g}_i, \mathfrak{g}_i] = \bigoplus_{i\in I} \mathfrak{g}_i = \mathfrak{g}\,.$$

□

4.4 Levi Decomposition

We see now how to put together semisimplicity and solvability in a general Lie group.

Lemma 4.11

If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals in a Lie algebra $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$ is a solvable ideal.

Proof The assertion is immediate from the short exact sequence

$$\{0\} \longrightarrow \mathfrak{a} \hookrightarrow \mathfrak{a} + \mathfrak{b} \longrightarrow (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}) \twoheadrightarrow \{0\}\,.$$

□

There is an analogous statement for nilpotent ideals. The proof requires some more work, but it is easy to convince oneself that the statement is true by looking at the effect of taking the bracket.

Remark. We can only write $\mathfrak{a} + \mathfrak{b}$ and not $\mathfrak{a} \oplus \mathfrak{b}$ as, a priori, $\mathfrak{a} \cap \mathfrak{b} \neq \{0\}$.

Corollary 4.14

For any Lie algebra $\mathfrak{g}$ there exists a unique maximal solvable ideal $\mathfrak{r} \subseteq \mathfrak{g}$ and $\mathfrak{g}/\mathfrak{r}$ is semisimple. Thus $\mathfrak{g}$ is semisimple if and only if $\mathfrak{r} = \{0\}$.

Definition 4.14. (Solvable) radical

The unique maximal solvable ideal of $\mathfrak{g}$ is called the (solvable) radical *of $\mathfrak{g}$.*

Proof [Proof of Corollary 4.14] The existence and uniqueness follow from Lemma 4.11 and from the finite dimensionality of $\mathfrak{g}$. To show that $\mathfrak{g}/\mathfrak{r}$ is semisimple, let $\mathfrak{h}/\mathfrak{r} \subseteq \mathfrak{g}/\mathfrak{r}$ be a solvable ideal in $\mathfrak{g}/\mathfrak{r}$. Then $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal and since $\mathfrak{r}$ and $\mathfrak{h}/\mathfrak{r}$ are solvable, it follows that $\mathfrak{h}$ is solvable. Since $\mathfrak{r}$ is maximal, then $\mathfrak{h} \subseteq \mathfrak{r}$, so that $\mathfrak{h}/\mathfrak{r} = \{0\}$ and $\mathfrak{g}/\mathfrak{r}$ has no solvable ideals. □

On the group level we have the following:

Corollary 4.15

Let G be a connected Lie group and R the connected subgroup corresponding to $\mathfrak{r} \subseteq \mathfrak{g}$. Then R is a solvable connected closed normal subgroup and G/R is a semisimple Lie group. ♡

Proof The only thing to show is that R is closed. Let $\overline{R}$ be the closure of R and let $\mathfrak{r}'$ be the corresponding Lie algebra. By maximality of $\mathfrak{r}$, it will be enough to show that $\mathfrak{r}'$ is solvable. Observe that if H is a connected Lie subgroup with $\mathrm{Lie}(H) = \mathfrak{h}$, then

$$\begin{aligned} H \text{ is solvable } &\Leftrightarrow \mathfrak{h} \text{ is solvable} \\ &\Leftrightarrow \mathrm{ad}(\mathfrak{h}^{\mathbb{C}}) = \mathrm{ad}(\mathfrak{h})^{\mathbb{C}} \text{ is upper triangular} \\ &\Leftrightarrow \mathrm{Ad}(H)^{\mathbb{C}} \text{ is upper triangular.} \end{aligned}$$

Thus, since R is solvable, $\mathrm{Ad}(R)^{\mathbb{C}}$ is upper triangular and, by continuity, $\mathrm{Ad}(\overline{R})^{\mathbb{C}}$ is upper triangular, that is $\overline{R}$ is solvable and $\mathfrak{r}'$ is solvable.

Remark Even though R is the maximal closed connected normal solvable subgroup, there might be larger solvable subgroups that are not connected.

For a general Lie algebra $\mathfrak{g}$ we have then the short exact sequence

$$\{0\} \longrightarrow \mathfrak{r} \hookrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{r} \twoheadrightarrow \{0\},$$

where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{r}$ is semisimple. It is natural to ask whether or not the sequence splits, that is whether or not we can write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, where $\mathfrak{r}$ is the solvable radical and $\mathfrak{s}$ is a semisimple subalgebra isomorphic to $\mathfrak{g}/\mathfrak{r}$.

Theorem 4.7. Levi Decomposition

Given any finite dimensional Lie algebra $\mathfrak{g}$ there exists a semisimple subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ as vector spaces and $\mathfrak{s} \simeq \mathfrak{g}/\mathfrak{r}$ as Lie algebras. Then $\mathfrak{s}$ is called the Levi subalgebra *or* Levi factor *or* semisimple factor *of $\mathfrak{g}$.*
The ideal $\mathfrak{r}$ is canonically determined by $\mathfrak{g}$, but if $\mathfrak{s}$ is a Levi subalgebra and $\phi \in \mathrm{Aut}(\mathfrak{g})$, then $\phi(\mathfrak{s})$ is another Levi subalgebra and any Levi subalgebra arises in this way. Moreover Levi subalgebras are maximal with respect to the property of being semisimple. ♡

To better understand Levi's theorem we need the notion of semidirect product of Lie algebras and Lie groups.

We saw in § 2.1 the definition of semidirect product of topological groups H, N. It is easy to see that if H, N are two Lie groups the semidirect product $H \ltimes N$ is the Lie group with the manifold structure of the product $H \times N$.

Definition 4.15

Let $\mathfrak{h}, \mathfrak{n}$ be two Lie algebras and $\rho\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$ a Lie algebra homomorphism. The semidirect product $\mathfrak{h} \ltimes_\rho \mathfrak{n}$ is the vector space $\mathfrak{h} \times \mathfrak{n}$ with bracket

$$[(H_1, N_1), (H_2, N_2)] := ([H_1, H_2], [N_1, N_2] + \rho(H_1)N_2 - \rho(H_2)N_1))$$

for all $H_1, H_2 \in \mathfrak{h}$ and all $N_1, N_2 \in \mathfrak{n}$ Thus $\mathfrak{h} \ltimes_\rho \mathfrak{n}$ is a Lie algebra (with Jacobi identity following from the fact that ρ is a Lie algebra homomorphism) and $\mathfrak{n}$ is an ideal in $\mathfrak{h} \ltimes_\rho \mathfrak{n}$. ♣

Parallel to Lemma 2.1, in the case of Lie algebras we have the following:

Lemma 4.12

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra and $\mathfrak{n} \subset \mathfrak{g}$ an ideal. The following are equivalent:

1. *There exists a Lie algebra homomorphism $\rho\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$ such that $\mathfrak{g} = \mathfrak{h} \ltimes_\rho \mathfrak{n}$;*
2. *$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$;*
3. *$\mathfrak{g}$ is a Lie algebra extension of $\mathfrak{n}$ by $\mathfrak{h}$, that is there exists a short exact sequence*

$$\{0\} \longrightarrow \mathfrak{n} \hookrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \twoheadrightarrow \{0\},$$

that splits, that is the composition $p \circ i\colon \mathfrak{h} \to \mathfrak{g}/\mathfrak{n}$ of the embedding $i\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$ and of the natural projection $p\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ is a Lie algebra isomorphism. ♡

It is easy to see that if $G = H \ltimes_\eta N$ is a semidirect product of Lie groups with $\eta\colon H \to \mathrm{Aut}(N)$ a smooth homomorphism, then $\mathfrak{g} = \mathfrak{h} \ltimes_\rho \mathfrak{n}$ is a semidirect product of the respective Lie algebras, where $\rho := d\eta_e\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$.

Hence Levi's Theorem says that $\mathfrak{g}$ is isomorphic as a vector space to $\mathfrak{s} \oplus \mathfrak{r}$ and as a Lie algebra to $\mathfrak{s} \ltimes_\rho \mathfrak{r}$ with respect to some homomorphism $\rho\colon \mathfrak{s} \to \mathrm{Der}(\mathfrak{r})$. Clearly changing the Levi subalgebra amounts to changing the homomorphism and there is no canonical Levi factor.

Example 4.8 Let $V \subset \mathbb{R}^n$ be a subspace and $\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{R}) : X(V) \subset V\} = \left\{\begin{pmatrix} *_V & * \\ 0 & * \end{pmatrix}\right\}$.
Let $\mathfrak{n} = \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\} \subset \mathfrak{g}$ and $\mathfrak{a} = \left\{\begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} \lambda_i \in \mathbb{R}\right\} \subset \mathfrak{g}$. Then $\mathfrak{n}$ is a nilpotent ideal in $\mathfrak{g}$ and $\mathfrak{a}$ is an Abelian subalgebra. We claim that $\mathfrak{a} + \mathfrak{n}$ is a solvable ideal. In fact one can check that $[\mathfrak{a}+\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{a}+\mathfrak{n}$, so $\mathfrak{a}+\mathfrak{n}$ is an ideal. Furthermore $[\mathfrak{a}+\mathfrak{n}, \mathfrak{a}+\mathfrak{n}] = [\mathfrak{a}, \mathfrak{n}] + [\mathfrak{a}, \mathfrak{a}] + [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$,

so $(\mathfrak{a}+\mathfrak{n})^{(1)} = [\mathfrak{a}+\mathfrak{n}, \mathfrak{a}+\mathfrak{n}] \subset \mathfrak{n}$ is nilpotent, and hence $\mathfrak{a}+\mathfrak{n}$ is solvable.

If $\dim V = k$, it is easy to see that $\mathfrak{g}/(\mathfrak{a}+\mathfrak{n})$ is isomorphic to the Lie algebra $\mathfrak{s} = \begin{pmatrix} \mathfrak{sl}(k,\mathbb{R}) & 0 \\ 0 & \mathfrak{sl}(n-k,\mathbb{R}) \end{pmatrix}$, hence it is semisimple. Thus $\mathfrak{a}+\mathfrak{n}$ is the solvable radical of $\mathfrak{g}$, $\mathfrak{s}$ is a Levi factor and $\mathfrak{g} = \mathfrak{s} \ltimes (\mathfrak{a}+\mathfrak{n})$.

Corollary 4.16

Let G be a simply connected Lie group and R its solvable radical. Then there exists a semisimple simply connected Lie subgroup S of G such that G, as a Lie group, is the semidirect product of S by R, $G = S \ltimes R$. All subgroups $S \leq G$ that may occur in this decomposition are isomorphic.

♡

Example 4.9 Let $G = \mathrm{GL}(n,\mathbb{R})$, $H = \mathbb{R}^n$, $\eta : G \to \mathrm{Aut}(\mathbb{R}^n)$ a smooth homomorphism and let us consider the group $G \ltimes_\eta \mathbb{R}^n =: G'$ of affine motions of $\mathbb{R}^n$. Then $G' = \begin{pmatrix} \eta(G) & \mathbb{R}^n \\ 0 & 1 \end{pmatrix}$, where $(A,v) \in G'$ acts on $\mathbb{R}^n$ by $x \mapsto \eta(A)x + v$, and the multiplication in G' is the composition of affine transformations.

Since $Z(G)$ consists of the group of scalar matrices, $G = \mathrm{GL}(n,\mathbb{R})$ is not semisimple and hence $G' = G \ltimes_\eta \mathbb{R}^n$ is not the Levi decomposition of G'. Let $A = \{\lambda \mathrm{Id} : \lambda \in \mathbb{R} \setminus \{0\}\} < \mathrm{GL}(n,\mathbb{R})$ and let us consider the group $R = A \ltimes_\eta \mathbb{R}^n < G \ltimes_\eta \mathbb{R}^n$. Then $\mathrm{Lie}(G'/R) = \mathfrak{sl}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$ is a Levi factor of G' and $A \ltimes \mathbb{R}^n$ is the radical of G'.

4.5 Compact Groups

Let V be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let G be a compact Lie group and let $\pi\colon G \to \mathrm{GL}(V)$ be a representation. Then π is equivalent to an orthogonal representation of G, that is there exists a positive-definite inner product $\langle\cdot,\cdot\rangle$ on V such that $\pi(G) \subseteq \mathrm{O}(V, \langle\cdot,\cdot\rangle)$. In fact, if $(\cdot,\cdot)$ is any inner product on V and μ is the Haar measure of G, then one can show that for $v, w \in V$

$$\langle v, w\rangle := \int_G (\pi(g)v, \pi(g)w)\, d\mu(g)$$

is a positive definite inner product on V that is $\pi(G)$-invariant by construction.

As a consequence we deduce the following:

Corollary 4.17

Any compact subgroup K of $\mathrm{GL}(n,\mathbb{R})$ is conjugate to a subgroup of $\mathrm{O}(n,\mathbb{R})$.

Proof Let $\pi = i\colon K \to \mathrm{GL}(n,\mathbb{R})$ be the inclusion. Then $K < \mathrm{O}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ and all orthogonal representations are conjugate to each other, so that a conjugate of K is contained in $\mathrm{O}(n,\mathbb{R})$. $\square$

Corollary 4.18

$O(n,\mathbb{R})$ is a maximal compact subgroup of $\mathrm{GL}(n,\mathbb{R})$ and it is unique up to conjugacy.

Lemma 4.13

Let G be a compact Lie group. Then $B_{\mathfrak{g}}$ is semidefinite negative. Moreover $B_{\mathfrak{g}}(X,X) = 0$ if and only if $\mathrm{ad}_{\mathfrak{g}}(X) = 0$.

Proof Since G is a compact connected semisimple Lie group, then $\mathrm{Ad}_G(G) < O(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) < \mathfrak{o}(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, that is elements of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ are skew-symmetric with respect to the G-invariant inner product $\langle\cdot,\cdot\rangle$. In other words if $X \in \mathfrak{g}$ and $A := \mathrm{ad}_{\mathfrak{g}}(X)$, then

$$B_{\mathfrak{g}}(X,X) = \mathrm{tr}(A^2) = \sum_{i,j} A_{ij}A_{ji} = -\sum_{i,j} |A_{ij}|^2 \leq 0$$

and $B_{\mathfrak{g}}(X,X) = 0$ if and only if $\mathrm{ad}_{\mathfrak{g}}(X) = 0$. But since $\mathfrak{g}$ is semisimple then $\ker(\mathrm{ad}_{\mathfrak{g}}) = \mathfrak{z}(\mathfrak{g}) = 0$, so $B_{\mathfrak{g}}(X,X) = 0$ if and only if $X = 0$, and $B_{\mathfrak{g}}$ is negative-definite. $\square$

Corollary 4.19

Let G be a connected semisimple Lie group. The following are equivalent:

1. *G is compact;*
2. *$B_{\mathfrak{g}}$ is negative-definite;*
3. *$B_{\mathfrak{g}}$ is definite.*

Proof $(1. \Rightarrow 2.)$ This is Lemma 4.13.

$(2. \Rightarrow 3.)$ Obvious.

$(3. \Rightarrow 1.)$ If $B_{\mathfrak{g}}$ is definite, then $O(\mathfrak{g}, B_{\mathfrak{g}})$ is a compact group. Since $\mathrm{Ad}_G(G) < O(\mathfrak{g}, B_{\mathfrak{g}})$, then $\mathrm{Ad}_G(G)$ is compact and semisimple (since G is semisimple). To conclude we need the following result, whose proof (of sketch thereof) we postpone to the end of the section.

Remark Notice that for the implication $(3. \Rightarrow 1.)$ we do not need the explicit hypothesis of semisimplicity of $\mathfrak{g}$, as it is automatically verified if $B_{\mathfrak{g}}$ is negative-definite, hence non-degenerate (Theorem 4.5).

Theorem 4.8

Let G be a compact semisimple Lie group. Then its universal cover $\widetilde{G}$ is also compact. Equivalently, $\pi_1(G)$ is finite. ♡

Applying Theorem 4.8, we conclude that the universal covering of $\mathrm{Ad}(G)(G)$ is compact. Since G is a covering of ${}_G(G)$, it has the same universal covering as $\mathrm{Ad}_G(G)$ and is hence compact. □

Proposition 4.13

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$, with $B_{\mathfrak{g}'}$ non-degenerate (so that $\mathfrak{g}'$ is semisimple). In fact, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$ and if $Z(G)$ is finite, then G is semisimple. ♠

Proof Obviously, if we assume that if $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$ with $\mathfrak{g}'$ semisimple, then

$$[\mathfrak{g}, \mathfrak{g}] = [Z(\mathfrak{g}) \oplus \mathfrak{g}', Z(\mathfrak{g}) \oplus \mathfrak{g}'] = [\mathfrak{g}', \mathfrak{g}'] = \mathfrak{g}'.$$

Moreover if $Z(G)$ is finite, then $Z(\mathfrak{g}) = 0$ and so $\mathfrak{g} = \mathfrak{g}'$ is semisimple.

To see that $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$, let $\mathfrak{g}' = \{X \in \mathfrak{g} : \langle X, Y\rangle = 0 \text{ for all } Y \in Z(\mathfrak{g})\} = Z(\mathfrak{g})^\perp$, where $\langle \cdot, \cdot\rangle$ is the G-invariant inner product on $\mathfrak{g}$ so that $\mathrm{Ad}_G(G) < \mathrm{O}(\mathfrak{g}, \langle \cdot, \cdot\rangle)$. Since $Z(\mathfrak{g})$ is an ideal in $\mathfrak{g}$, then the same proof as for Lemma 4.6 (where we use the invariance of $\langle \cdot, \cdot\rangle$) shows that $Z(\mathfrak{g})^\perp = \mathfrak{g}'$ is also an ideal and $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$.

The only thing to show is that $\mathfrak{g}'$ is semisimple, or equivalently that $B_{\mathfrak{g}'}$ is non-degenerate. Since $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, $B_{\mathfrak{g}'} = B_{\mathfrak{g}}|_{\mathfrak{g}' \times \mathfrak{g}'}$ (Lemma 4.5). This is non-degenerate, because as in the proof of Corollary 4.19 we have that $B_{\mathfrak{g}}(X, X) = 0$ if and only if $\mathrm{ad}_{\mathfrak{g}}(X) = 0$, if and only if $X \in Z(\mathfrak{g})$. So $\mathfrak{g}'$ is semisimple. □

Corollary 4.20

Let G be a compact connected Lie group. Then $G = TK$, where T and K are closed connected normal subgroups, $T < Z(G)$, K is compact and semisimple and $T \cap K$ is finite. ♡

Remark TK is an almost direct product, that is each element in G can be written as a product of an element in T and an element in K but not uniquely (although only in finitely many ways). However the product is still well-defined, since T is central, hence $t_1 k_1 t_2 k_2 = t_1 t_2 k_1 k_2$ for all $t_1, t_2 \in T$ and all $k_1, k_2 \in K$.

Proof By proposition 4.13, $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$, where $B_{\mathfrak{g}'}$ is non-degenerate. Since $b_{\mathfrak{g}}|_{\mathfrak{g}' \times \mathfrak{g}'} = B_{\mathfrak{g}'}$

and $B_{\mathfrak{g}}$ is seminegative definite, this implies that $B_{\mathfrak{g}'}$ is negative definite and hence (Corollary **??**) the connected Lie subgroup $K < G$ associated to $\mathfrak{g}'$ is compact. Let $T = Z(G)^\circ$, so that $\mathrm{Lie}(T) = Z(\mathfrak{g})$. THe map

$$\begin{aligned} K \times T &\longrightarrow G \\ (k,t) &\longmapsto kt \end{aligned} \tag{4.7}$$

is a Lie group homomorphism whose derivative at (e,e) is the map

$$\begin{aligned} \mathfrak{g}' \times Z(\mathfrak{g}) &\longrightarrow \mathfrak{g} \\ (X,Y) &\longmapsto X+Y\,, \end{aligned} \tag{4.8}$$

which is a Lie algebra homomorphism. In particular KT contains an open neighborhood of $e \in G$, hence KT is an open subgroup of G. Since G is connected, this implies that $KT = G$. Finally, $K \cap T < Z(K)$ and the latter is finite since K is compact semisimple. □

Proof [Sketch of the Proof of Theorem 4.8] The fundamental group of a compact manifold is finitely generated and the fundamental group of a topological group is Abelian. Thus by the classification theorem for finitely generated Abelian groups, we have that

$$\pi_1(G) \simeq \bigoplus_{i=1}^{\ell} \mathbb{Z} \oplus \bigoplus_{j=1}^{q} \mathbb{Z}/n_j\mathbb{Z}\,,$$

and we want to show that $\ell = 0$. If $H_1(M,\mathbb{Z})$ denotes the singular homology of a manifold M with integer coefficients, we have

$$\pi_1(M)/[\pi_1(M),\pi_1(M)] \simeq H_1(M,\mathbb{Z})\,,$$

but since G is a topological group and $\pi_1(G)$ is Abelian, then $[\pi_1(G),\pi_1(G)] = \{e\}$ and hence

$$\pi_1(G) \simeq H_1(G,\mathbb{Z})\,.$$

By the Universal Coefficients Theorem

$$H^1(G,\mathbb{R}) \simeq \mathrm{Hom}(H_1(G,\mathbb{Z}),\mathbb{R}) = \mathrm{Hom}(\pi_1(G),\mathbb{R}) = \mathrm{Hom}\left(\bigoplus_{i=1}^{\ell} \mathbb{Z} \oplus \bigoplus_{j=1}^{q} \mathbb{Z}/n_j\mathbb{Z}, \mathbb{R}\right) \simeq \mathbb{R}^\ell\,,$$

so that it will be enough to show that $H^1(G,\mathbb{R}) = 0$.

Recall that there is an isomorphism $H^*(G,\mathbb{R}) \simeq H^*_{dR}(G)$ of the group cohomology with the de Rham cohomology of G. Recall moreover that:

1. If ω is a 1-form on a manifold M, $d\omega$ is the 2-form defined by

$$d\omega(X,Y) := X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\,,$$

where $X, Y \in \mathrm{Vect}(M)$.

2. (Cartan) If G is a compact connected Lie group, then $H^*_{dR}(G)$ is isomorphic to the homology of complex $\Omega^*(G)^G$ of G-invariant differential forms on G.

Thus to show that $H^*(G, \mathbb{R}) = 0$ it will be enough to show that if ω is closed (that is $d\omega = 0$), then $\omega = 0$.

Let $X_e, Y_e \in T_eG$ and let $X, Y \in \mathrm{Vect}(G)^G$ the corresponding G-invariant vector fields. By invariance of ω and of X, Y, we have that $\omega(X)$ and $\omega(Y)$ are both constant, so that

$$0 = d\omega(X, Y)_e = (X(\omega(Y)))_e - (Y(\omega(X)))_e - (\omega([X, Y]))_e = 0 - 0 + (\omega([X, Y]))_e\,.$$

Since $\mathfrak{g}$ is semisimple and hence $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, from $d\omega = 0$ follows that $\omega = 0$. $\square$

Chapter 4 Exercise

1. Prove Lie Theorem for Lie algebras.
2. If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}^{\mathbb{C}}$ is nilpotent.

Chapter 5 Lie Groups as Transformation Groups

Appendix Preliminaries

p.131 (paragraph before definition A.11) The explanation for the coordinate chart of the tangent bundle is, if I am not mistaken, not correct. At least the definition that I know, is the one where the isomorphism $\mathbb{R}^n \to T_pM$ isn't any isomorphism, but the one induced by the coordinate chart (ϕ, U) that is $d\phi$.

A.1 Topological Preliminaries

We recall now a few well known concepts from topology.

Definition A.1. Basis of a topology

A basis $\mathcal{B}$ *of a topology* $\mathcal{T} \subset \mathcal{P}(X)$ *on a set* X *is a family* $\mathcal{B} \subset \mathcal{T}$ *such that every element of* $\mathcal{T}$ *is the union of elements of* $\mathcal{B}$. ♣

Example A.1 The family

$$\mathcal{B} := \{B_r(x) : r \in \mathbb{Q}_{\geq 0},\, x \in \mathbb{Q}^n\}$$

is a basis of the Euclidean topology on $\mathbb{R}^n$.

Lemma A.1. Characterization of a basis

Let X *be a set and* $\mathcal{T} \subset \mathcal{P}(X)$ *a topology. A family* $\mathcal{B} \subset \mathcal{T}$ *is a basis if and only if*

- $X = \cup_{Y \in \mathcal{B}} Y$*, and*
- *If* $B_1, B_2 \in \mathcal{B}$ *and* $B_1 \cap B_2 \neq \emptyset$*, then for every* $x \in B_1 \cap B_2$ *there exists* $B_3 \in \mathcal{B}$ *with* $x \in B_3 \subset B_1 \cap B_2$.

Then the topology is the family consisting of all possible unions of elements in $\mathcal{B}$. ♡

Definition A.2. Subbasis

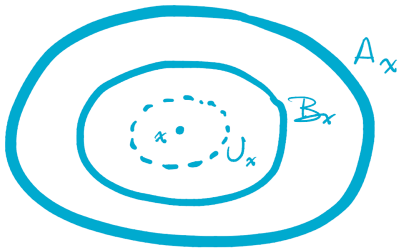
A subbasis $\mathcal{S}$ *of a topology* $\mathcal{T} \subset \mathcal{P}(X)$ *on a set* X *is a family of sets such that the family* $\mathcal{B}$ *obtained by taking all finite intersections of elements in* $\mathcal{S}$ *is a basis.* ♣

Definition A.3. Hausdorff topology

A topological space X is Hausdorff *if any two distinct points have disjoint neighborhood.* ♣

Definition A.4. Local Compactness

A topological space X is locally compact *if each point has a neighborhood basis consisting of compact sets, that is if for every $x \in X$ there exists a set $\mathcal{B}_x$ of compact neighborhoods of x such that any neighborhood A_x of x contains an element $B_x \in \mathcal{B}_x$.* ♣



Lemma A.2

Let X be a locally compact Hausdorff topological space. Every closed subset and every open subset of X is locally compact with respect to the induced topology. ♡

For any topological spaces X, Y one can define different topologies on the set

$$Y^X := \{f\colon X \to Y\}\,,$$

or more specifically on the set

$$C(X,Y) : \{f\colon X \to Y :\ f \text{ is continuous}\}\,.$$

Definition A.5

Let X, Y be topological spaces.

- *The sets*

$$S(C,U) := \{f \in C(X,Y), f(C) \subset U\}$$

where $C \subset X$ is a compact set and $U \subset Y$ is an open set, form a subbasis of the compact-open *topology on $C(X,Y)$.*

- *The sets*

$$S(x,U) := \{f \in C(X,Y) :\ f(x) \in U\}$$

form a subbasis of the topology of the pointwise open *(or* pointwise convergence*) topology on $C(X,Y)$* ♣

Remark Let X be a topological space and (Y, d) a metric space. The sets

$$B_C(f, \epsilon) := \{g \in C(X, Y) : \sup_{x \in C} d(f(x), g(x)) < \epsilon\},$$

where $C \subset X$ is a compact set, $\epsilon > 0$ and $f \in C(X, Y)$ form a basis of the compact-open topology. The set $B_C(f, \epsilon)$ consists of all functions $g \in C(X, Y)$ that are ϵ-close to f in all points in the compact set C. It is easy to see that if $\{f_n\} \subset C(X, Y)$, then $f_n \to f$ in the compact-open topology if and only if $f_n|_C \to f|_C$ uniformly on all compact sets $C \subset X$. In other words, if Y is a metric space the compact-open topology is nothing but the topology of the *uniform convergence on compact sets*.

In general the pointwise convergence is weaker than the uniform convergence on compact sets, which, in turn, is weaker that the uniform convergence. Of course the first two coincide on a set with the discrete topology and the last two on a compact set.

A.2 Functional Analytical Preliminaries

Theorem A.1. Ascoli–Arzelà's Theorem

Let (X, d_X) and (Y, d_Y) be compact metric spaces and let us consider the Banach space $C(X, Y)$ of continuous functions $f : X \to Y$ with the metric

$$d(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

Let $\mathcal{F} \subset C(X, Y)$ be a subfamily of continuous functions. Then $\mathcal{F}$ is relatively compact if and only if it is equicontinuous, *that is for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) < \varepsilon$$

for every $f \in \mathcal{F}$, whenever $d_X(x, y) < \delta$. ♡

This is the form of the theorem that we need. Notice however that

- X need not be a metric space for the theorem to hold, and
- If Y is not compact then the theorem still holds, provided we add the assumption that the set $\{f(x) : f \in \mathcal{F}\}$ i s relatively compact for all $x \in X$.

If E, F are normed spaces, let us consider the normed space

$$\mathcal{B}(E, F) := \{T : E \to F : T \text{ is continuous and linear }\},$$

with $\|T\| := \sup_{\|x\|_E=1} \|T(x)\|_F$.

If $T \in \mathcal{B}(E, F)$ is bijective and the inverse is continuous, then T is an isomorphism of E with F. If in particular $E = F$, then T is an automorphism of E, and we denote $\mathrm{Aut}(E) \subset \mathcal{B}(E)$ the subspace of automorphisms. If in particular E is of finite dimension n, then $\mathrm{Aut}(E) = \mathrm{GL}(E)$.

Definition A.6. Topologies on $\mathcal{B}(E, F)$

Let $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(E, F)$.

1. *We say that* $T_n \to T$ *in the* norm topology *if and only if* $\lim_{n\to\infty} \|T_n - T\| = 0$, *where* $\|\cdot\|$ *is the norm on* $\mathcal{B}(E, F)$.
2. *We say that* $T_n \to T$ *in the* strong operator topology *if and only if* $\lim_{n\to\infty} \|T_n x - Tx\|_F = 0$ *for all* $x \in E$.
3. *We say that* $T_n \to T$ *in the* weak operator topology *if* $\lim_{n\to\infty} \lambda(T_n x) = \lambda(Tx)$ *for all* $\lambda \in F^*$.

♣

In particular if E is a normed vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$ and $F = k$, then $\mathcal{B}(E, k)$ is nothing but the dual E^* of E and the strong operator topology on $\mathcal{B}(E, k)$ is nothing but the weak-$*$-topology on E^*.

If $\mathcal{H}$ is a Hilbert space and $E = F = \mathcal{H}$, then the space of isometric isomorphisms of E $\mathrm{Iso}(E)$ is the space of unitary operators $\mathcal{U}(\mathcal{H})$. On $\mathcal{U}(\mathcal{H})$ the strong operator topology and the weak operator topology coincide.

Let G be a topological group and E a topological vector space. A *continuous representation of G on E* is a homomorphism $\pi\colon G \to \mathrm{Aut}(E)$, which is continuous with respect to a topology on $\mathrm{Aut}(E)$. If in particular, E is a normed space, then π is an *isometric representation* if $\pi\colon G \to \mathrm{Iso}(E)$. An isometric representation of a Hilbert space is called *unitary*.

Lemma A.3

Let G be a topological group acting continuously on a locally compact space X. Let $C_c(X)$ be the space of continuous functions with compact support on X with the norm topology. Then the representation $\pi\colon G \to \mathrm{Iso}(C_c(X))$ defined by

$$\pi(g)f(x) := f(g^{-1}x)$$

for $x \in X$ and $g \in G$ is a continuous representation if $\mathrm{Iso}(C_c(X))$ is endowed with the strong operator topology.

♡

If E, F are topological vector spaces and $T \in \mathcal{B}(E, F)$, the *adjoint* $T^* : F^* \to E^*$ is defined by

$$T^*(\lambda) := \lambda \circ T\,.$$

In particular, if E is a topological vector space on which G acts via a representation π, and E^* is endowed with the weak-$*$-topology, then

$$\pi^*(g) := \pi(g^{-1})^* \colon E^* \to E^*$$

is continuous.

Definition A.7. regular Borel measure

1. *Let X be a locally compact Hausdorff space. A measure on the σ-algebra of Borel sets of X is called a* Borel measure *if it is finite on every compact set.*
2. *A Borel measure μ is said to be* regular *if*
   (a). *for every Borel set Y, $\mu(Y) = \sup \mu(K)$ over all compact subsets $K \subseteq Y$, and*
   (b). *for every σ-bounded set Y, $\mu(Y) = \inf \mu(U)$ over all open σ-bounded sets $U \supseteq Y$ for every set U in B(X).*

♣

Recall that a set Y is σ-bounded if it is contained in the countable union of compact sets.

Definition A.8. Separability

Let $\mathcal{H}$ be a complex Hilbert space. We say that $\mathcal{H}$ is separable if it contains a countable dense subset. ♣

A.3 Differentiable Manifolds

Definition A.9. Paracompactness

A topological space X is paracompact if every open covering $\{U_\alpha\}_{\alpha\in A}$ has a locally finite refinement, that is there exists a covering $\{V_\beta\}_{\beta\in B}$ such that

- *For every $\beta \in B$ there exists at least one $\alpha \in A$ such that $V_\beta \subset U_\alpha$, and*
- *for every $p \in X$ there exists a neighborhood W of x that intersects finitely many V_β.* ♣

For us a *smooth manifold* will always be Hausdorff, locally Euclidean with countable basis and paracompact.

Definition A.10. Germs

Given $p \in M$, we denote by $C^\infty(p)$ the algebra of germs *of smooth functions at p. This is the algebra of smooth functions defined in an open neighborhood of p, where two functions are identified if they coincide on a neighborhood of p.* ♣

Recall that the *tangent space* T_pM to the manifold M at the point p is the set of all linear functionals $X_p\colon C^\infty(p) \to \mathbb{R}$ such that for all $\alpha, \beta \in \mathbb{R}$ and all $f, g \in C^\infty(p)$:

1. $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$ (linearity);
2. $X_p(fg) = X_p(f) \cdot g(p) + f(p) X_p(g)$ (Leibniz rule).

The linear map $X_p \in T_pM$ is called a *tangent vector* to M at p and the tangent space T_pM has the structure of real vector space with operations:

1. $(X_p + Y_p)(f) := X_p(f) + Y_p(f)$;
2. $(\alpha X_p)(f) := \alpha X_p(f)$.

Let $f\colon M \to N$ be a smooth map of smooth manifolds and let $p \in M$. The *differential of f at p* is the linear map $d_pf\colon T_pM \to T_{f(p)}N$ defined as follows: if $X_p \in T_pM$ and $\phi \in C^\infty(f(p))$, then

$$d_pf(X_p) := X_p(\phi \circ f)\,.$$

In other words, the tangent vector $d_pf(X_p)$ applied to the function ϕ takes the derivative of the function $\phi \circ f$ at the point $p \in M$ in the direction of the tangent vector X_p.

The *tangent bundle* to M is $TM = \bigcup_{p \in M} T_pM$. It can be made into a manifold with coordinate charts $(U \times \mathbb{R}^n, \varphi \times \psi)$, where (U, φ) is a coordinate chart on M and $\psi\colon \mathbb{R}^n \to T_pM$ is an isomorphism. With this smooth structure the projection $\pi\colon TM \to M$ is smooth.

Definition A.11. Smooth vector field

A smooth vector field *is smooth section of the tangent bundle*

$$X\colon M \to TM$$

$\pi \circ X = id_M$. In other words, it is a map

$$\begin{aligned} X\colon M &\to TM \\ p &\mapsto X_p \in T_pM \end{aligned}$$

that assigns to each point $p \in M$ a tangent vector X_p to M at p, and such that the map

$$\begin{aligned} Xf \colon M &\to \mathbb{R} \\ p &\mapsto X_p(f) \end{aligned}$$

is smooth, for every $f \in C^\infty(M)$. ♣

It can be proven that if $p \in M$, then

$$X_p(f) = d_p f(X_p)\,, \tag{A.1}$$

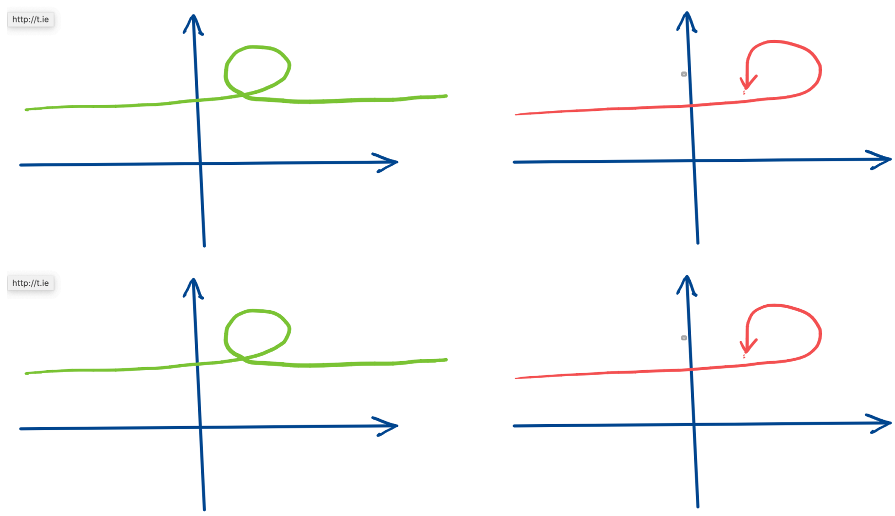
that is $X_p(f)$ is the differential of the function f at the point p in the direction of X_p.

Definition A.12

Let $\varphi : M \to N$ be a smooth map of smooth manifolds. Then:

1. *φ is an* immersion *if $d_p\varphi$ is non-singular for all $p \in M$.*
2. *$\varphi(M)$ is a* submanifold *or an* immersed sumbanifold *of N if φ is a one-to-one immersion.*
3. *If φ is a one-to-one immersion that is also a a homeomorphism of M onto its image, then φ is an embedding and $\varphi(M)$ is an embedded submanifold.* ♣

In the following pictures in green we see two immersion and in red two immersed submanifolds.

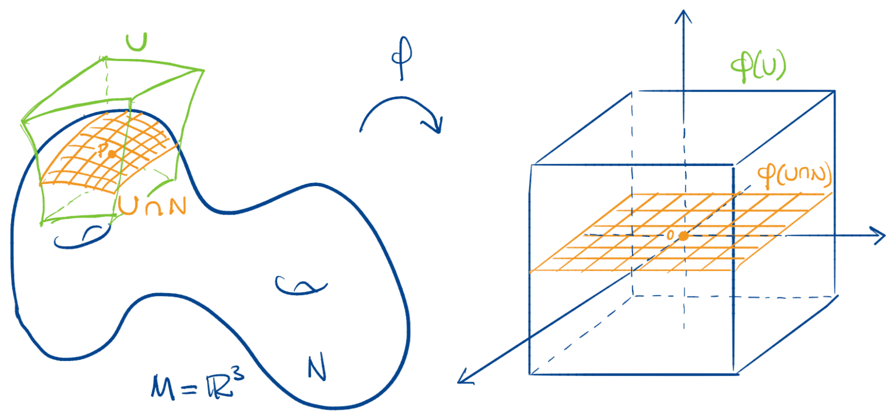


An embedded submanifold has the smooth structure coming from the ambient manifold and the concept of embedded submanifold are essentially equivalent to that of regular submanifold that we recall now.

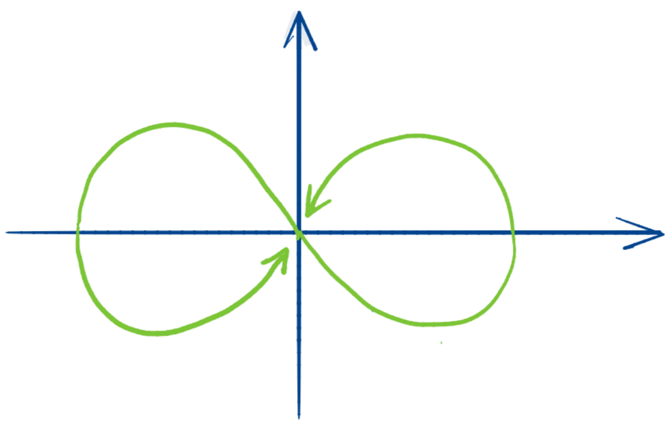
Definition A.13. (Regular Submanifold)

Let M be a smooth m-dimensional manifold.

1. *A subset $N \subset M$ has the* submanifold property *if every $p \in N$ has a coordinate neighborhood (U, φ) in M with local coordinates $x_1, \ldots, x_m$ such that*
   (a). $\varphi(p) = 0$;
   (b). $\varphi(U)$ *is an open cube* $(-\varepsilon, \varepsilon)^m$ *of side length* 2ε;
   (c). $\varphi(U \cap N) = \{x \in (-\varepsilon, \varepsilon)^m : x_{n+1} = \cdots x_m = 0\}$.
2. *A* regular submanifold *of M is any subset $N \subset M$ with the submanifold property and the smooth structure determined by the coordinate neighborhoods defined by the submanifold property.* ♣



Example A.2 The following is not a regular submanifold of $\mathbb{R}^2$.



The point of a regular submanifold is that the topology and the differentiable structure are those derived from M.

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