1. Introduction.

The origins of Lie theory are geometric and stem from Felix Klein's (1849-1925) so called Erlangen programm that the geometry of a space is determined by its group of symmetries.
In practice the geometry M and its group of symmetries G have a manifold structure and moreover the action $\alpha : G \times M \to M$ is smooth. This way every tangent vector $X \in T_e M$ gives rise to a smooth vector field on M :

pick a point $p \in M$, look at the derivative at $e \in G$ of the action map $\alpha_p : G \longrightarrow M$, $g \mapsto g.p$ and take the image of X to obtain a tangent vector $D_e \alpha_p(X) \in T_p M$.

This is $\underbrace{\text{Lie's}}_{(1842-1899)}$ viewpoint to look at the action infinitesimally. This way one obtains a finite dimensional vector space of smooth vector fields that is closed under the bracket operation: that is a Lie algebra. The smooth action is thus encoded infinitesimally by an algebraic object called Lie algebra.

It is Wilhelm Killing (1847-1923) who

insisted that one should thus classify ~~and~~ all finite dimensional Lie algebras.

The notion of Lie algebra of a Lie group G is obtained by applying the above construction to G acting by right translations on itself.

The structure of the course is as follows:

Chapter 2: Topological groups; homogeneous spaces and invariant measures.

Chapter 3: Lie groups and their Lie algebras: correspondence between subalgebras and subgroups.

Chapter 4: Structure theory.

In structure theory we will see that a general Lie algebra can be decomposed into two pieces, one a solvable algebra, the other semisimple. Semisimple algebras were eventually classified by E. Cartan (1869 - 1951) in terms of combinatorial data.

2. Topological groups.

2.1. Definitions and examples.

__Def. 2.1__ A topological group is a group G whose underlying set is endowed with a topology such that the multiplication

$$m : G \times G \to G$$
$$(g, h) \mapsto g \cdot h$$

and the inverse

$$i : G \to G$$
$$g \mapsto g^{-1}$$

are continuous maps.

In the above definition $G \times G$ is endowed with the product topology.

Elementary consequences 2.2.

(i) Since the inverse $i: G \to G$ is continuous and $i \circ i = \mathrm{id}$, it follows that i is a homeomorphism.

(ii) For every $g \in G$, the left translation

$$L_g : G \to G, \quad x \mapsto gx$$

and the right translation

$$R_g : G \to G, \quad x \mapsto x \cdot g$$

are continuous. Moreover,

$$L_g \circ L_{g^{-1}} = L_{g^{-1}} \circ L_g = \mathrm{id}, \quad R_g \circ R_{g^{-1}} = R_{g^{-1}} \circ R_g = \mathrm{id}$$

and hence L_g and R_g are homeomorphisms.

(iii) Let $\varphi : G_1 \to G_2$ be a homomorphism of top. groups. The condition

$$\varphi(g \cdot h) = \varphi(g)\,\varphi(h) \quad \forall g, h \in G_1$$

can also be equivalently expressed by the commutativity of the diagram

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] & G_2 \arrow[d, "L_{\varphi(g)}"] \\ G_1 \arrow[u, "L_{g^{-1}}"] \arrow[r, "\varphi"] & G_2 \end{tikzcd} \qquad \forall g \in G_1$$

Which implies that if φ is continuous at e, it is continuous at every $g \in G_1$.

(iv) A subgroup $H \leq G$ of a topological group G is a topological group when equipped with the induced topology.

(v) Given topological groups G_α, $\alpha \in A$ the cartesian product $\prod_{\alpha \in A} G_\alpha$ with product topology is a top. group.

(VI) $H \lhd G$ a normal subgroup of a topological group then the quotient $G/_H$ with quotient topology is a topological group. We will say more about quotients in section 2.5.2 on homogeneous spaces.

Let us turn to examples.

<u>Example 2.3</u> : any group G with the discrete topology.

<u>Example 2.4</u> : $\mathbb{R}^n$ with euclidean topology and addition $+$.

<u>Example 2.5</u> The multiplicative groups $\mathbb{R}^\times$ and $\mathbb{C}^\times$ of the fields $\mathbb{R}$ and $\mathbb{C}$.

<u>Example 2.6</u>. Let $M_{n,n}(\mathbb{R})$ be the space of $n \times n$ real matrices with euclidean topology. Then the matrix product $(A, B) \longmapsto A \cdot B$ is clearly continuous. The subset

$$GL(n, \mathbb{R}) := \{ A \in M_{n,n}(\mathbb{R}) : \det A \neq 0 \}$$

is open; it is a group with neutral element Id, as the inverse of A is given by

$$(A^{-1})_{ij} = \frac{\det M_{ij}}{\det A}$$

where M_{ij} is the cofactor matrix.

With this we see that $GL(n, \mathbb{R})$ is a topological group.

Given a locally compact Hausdorff space X it is natural to endow its group of homeomorphisms $Homeo(X)$ with the compact open topology. It ~~is~~ is then <u>not</u> necessarily a topological group, see chapter 2 exercise 8 b.) in Iozzi's notes. It is so if X is compact, see 8 a.).

<u>Example 2.7</u>

If X locally compact Hausdorff and locally connected, then $Homeo(X)$ is a top. group. This holds for instance if X is a manifold.

Another important source of top. groups is:

Example 2.8 Let (X,d) be a proper metric space, that is one in which all closed balls of finite radius are compact. Then its group of isometries

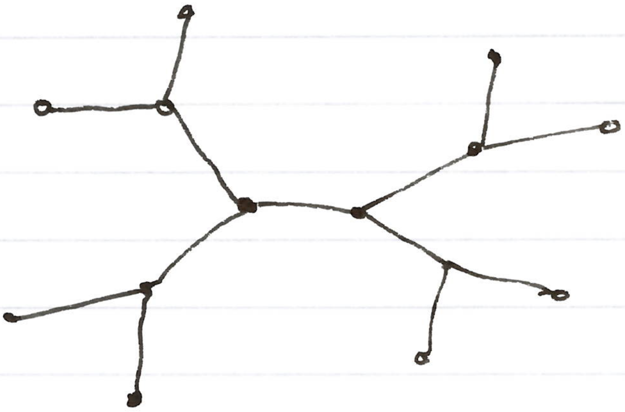
$$\mathrm{Iso}(X) := \{ f : X \to X : \text{bijection with } d(f(x), f(y)) = d(x,y) \ \forall x, y \in X \}$$

with compact open topology is a topological group.

For example if $X = \mathbb{R}^n$ with euclidean distance $\mathrm{Iso}(X) = O(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

We may also choose X to be the set of vertices of a regular tree T_d of valency d withe the combinatorsal distance.

T_3 :



While $\mathrm{Iso}(T_d)$, $d \geq 3$ is not related to any Lie group in any sense, the study of its closed subgroups is a very active domain of research.

Example 2.9 : The ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and its fraction field $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$. Then $\mathbb{Q}_p$ is a locally compact non-discrete field. (See Example 2.10 in Iozzi's notes).

<u>Example 2.10</u> The following subgroups of the topological group $GL(n, \mathbb{R})$ ~~wit~~ play an important role in its structure theory.

(i) $$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}_+^\times \right\}$$

is a closed subgroup isomorphic to $(\mathbb{R}_+^\times)^n$ as topological group.

(ii) The group of strictly upper triangular matrices

$$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : * \in \mathbb{R} \right\}$$

is a closed subgroup of $GL(n, \mathbb{R})$; it is isomorphic to $(\mathbb{R}, +)$ for $n = 2$ and non-abelian for $n \geq 3$.

(iii) The group $K = O(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) : A \cdot {}^tA = \mathrm{Id} \}$ of orthogonal matrices is a closed subgroup of $GL(n, \mathbb{R})$.

Notice that the Gram-Schmidt orthogonalization procedure can be formulated as follows: every $g \in GL(n, \mathbb{R})$ can be uniquely written as

$$g = k \cdot a \cdot n$$

with $k \in O(n, \mathbb{R})$, $a \in A$, $n \in N$. In fact the product map

$$K \times A \times N \longrightarrow GL(n, \mathbb{R})$$

$$(k, a, n) \longmapsto k \cdot a \cdot n$$

is a homeomorphism. In general Lie theory this is the Iwasawa decomposition.

Linear algebra provides us with a plethora of topological groups, as the following examples show:

<u>Example 2.11</u> ~~Let~~ Consider the bilinear symmetric form on $\mathbb{R}^n$:

$$B(x,y) := -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n} x_j y_j$$

$$= {}^t x \begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix} y$$

where $n = p+q$. This is the symmetric non-degenerate bilinear form of signature (p,q). The group $O(p,q)$

$= \{ g \in GL(n,\mathbb{R}) : g \text{ preserves } B \}$

$= \left\{ g \in GL(n,\mathbb{R}) : {}^t g \begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix} g = \begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix} \right\}$

is a closed subgroup of $GL(n, \mathbb{R})$.

<u>Example 2.12</u> As in Example 2.6 one defines

$$GL(n, \mathbb{C}) = \{ X \in M_{n,n}(\mathbb{C}) : \det X \neq 0 \}.$$

It is an open subset of the finite dimensional $\mathbb{C}$-vector space $M_{n,n}(\mathbb{C})$ and with induced topology it becomes a topological group.

Let for $X \in M_{n,n}(\mathbb{C})$ define $X^* := {}^t\overline{X}$. Then we define as in 2.10:

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}_+^* \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : * \in \mathbb{C} \right\}$$

$$U(n) := \{ g \in GL(n,\mathbb{C}) : gg^* = \mathrm{Id} \}$$

the latter being the unitary group.

Then the Iwasawa decomposition

$$K \times A \times N \longrightarrow GL(n,\mathbb{C})$$
$$(k, a, n) \longmapsto k \cdot a \cdot n$$

holds as well.

<u>Example 2.13</u> Very often one restricts to matrices of determinant 1; this leads then to closed subgroups of the linear groups defined so far.

Namely: $SL(n,\mathbb{R}) = \{ g \in GL(n,\mathbb{R}) : \det g = 1 \}$

$$SO(p,q) = O(p,q) \cap SL(n,\mathbb{R})$$

$$SL(n,\mathbb{C}) = \{ g \in GL(n,\mathbb{C}) : \det g = 1 \}$$

$$SU(n) = U(n) \cap SL(n,\mathbb{C}).$$

2.2. Compactness and Local Compactness.

Local compactness imposes a sharp divide between top. groups that are and the others, mainly due to the existence of Haar measure in the former case. Recall:

Def. 2.14 a topological space X is locally compact if every point admits a compact neighborhood.

For us the salient fact will be

Fact 2.15 : a Hausdorff space X is locally compact iff every point admits a fundamental system of compact neighborhoods.

Of course not every subspace of a locally compact space is locally compact. The following criterion is very useful

<u>Fact 2.16</u> Let X be l.c. Hausdorff. Then $Y \subset X$ is l.c. iff it is open in its closure.

With this at hand we are going to analyze our examples so far and add some new ones.

Clearly G with discrete topology (2.3); $(\mathbb{R}^n, +)$ (2.4); $\mathbb{R}^\times$, $\mathbb{C}^\times$ (2.5) are locally compact Hausdorff. So is (2.6) $GL(n, \mathbb{R})$ since it is an open subset of $M_{n,n}(\mathbb{R})$; In Example 2.7

if M is a manifold with $\dim M \geq 1$ then $\mathrm{Homeo}(M)$ is not l.c. (exercise)

Example 2.17 Let G_α, $\alpha \in A$ be a set of Hausdorff topological groups. Then

(i) $\prod_{\alpha \in A} G_\alpha$ is compact iff G_α is compact $\forall \alpha \in A$.

(ii) $\prod_{\alpha \in A} G_\alpha$ is locally compact iff all G_α's are l.c. AND all G_α are compact except finitely many.

Example 2.18 (see Example 2.8)

If (X,d) is a proper metric space then $\mathrm{Iso}(X)$ is locally compact and if X is compact, $\mathrm{Iso}(X)$ is also.

All this follows from the following version of Arzela - Ascoli : if (X, d) is any metric space then $\mathcal{F} \subset C(X, X)$ (continuous maps $X \to X$) has compact closure if $\mathcal{F}$ is equicontinuous and for every $x \in X$, the set $\{ f(x) : f \in \mathcal{F} \}$ has compact closure.

It is instructive to look more closely at the locally compact group $Iso(T_d)$. (exercise).

<u>Example 2.19</u> A, N and K are closed subgroups of $GL(n, \mathbb{R})$ hence locally compact. In addition

We claim that $O(n)$ is compact.

Indeed $O(n)$ is the inverse image of Id by the continuous map

$$M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$$
$$A \longmapsto A^t A$$

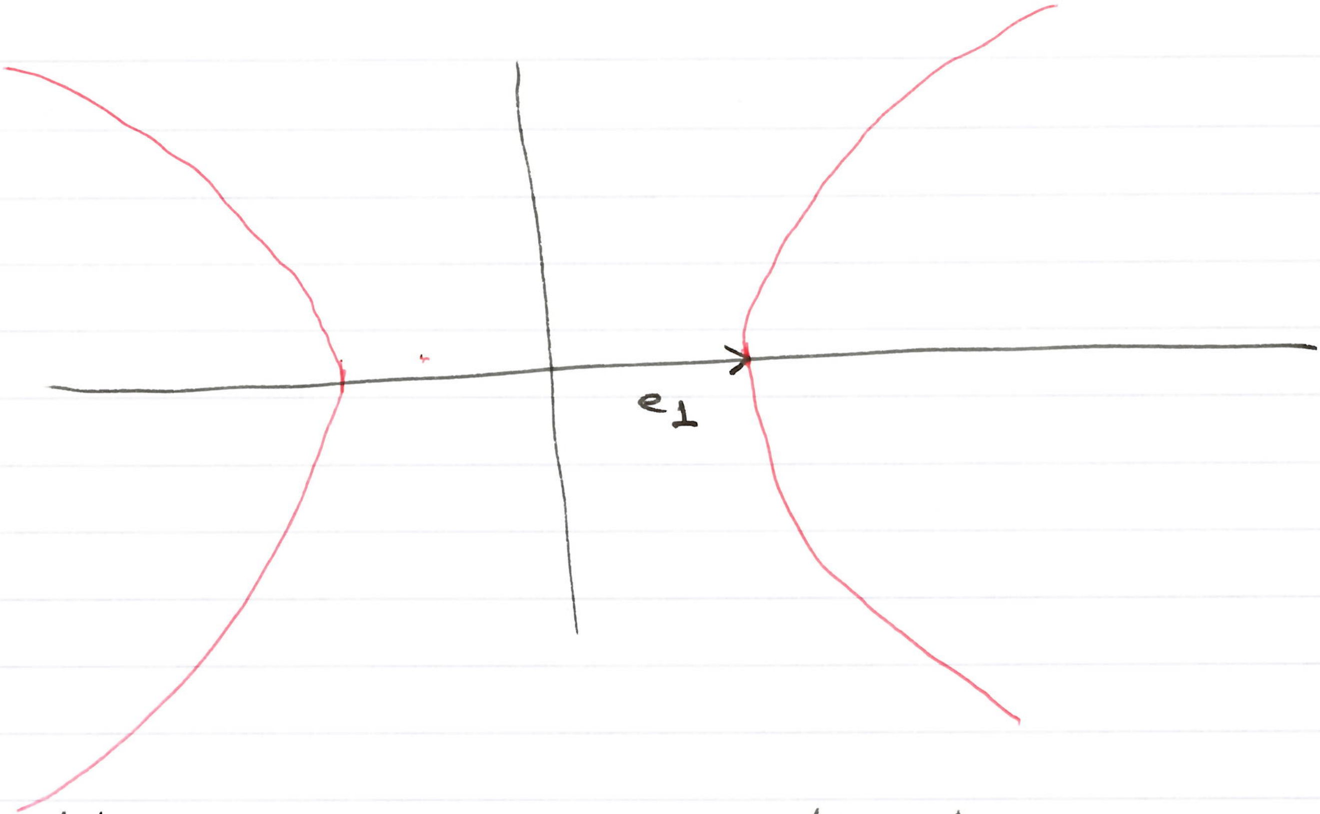
hence closed in $M_{n,n}(\mathbb{R})$. Writing a matrix X as $X = (X_1, \dots, X_n)$, X_i column vectors the condition $A^t A = Id$ implies $\| X_i \|^2 = 1$, hence $\sum_{i=1}^{n} \| X_i \|^2 = n$ which implies $O(n)$ is a bounded subset of $M_{n,n}(\mathbb{R})$; being also closed this implies compactness by Heine-Borel.

Example 2.20

We mentionned (Ex. 2.18) that $O(p,q)$ is a closed subgroup of $GL(n,\mathbb{R})$. When either $p=n$ or $q=n$ we have $O(n,0) = O(n,\mathbb{R}) = O(0,n)$ hence is compact. In all other case we have $p \geqslant 1$ and $q \geqslant 1$, and we claim that $O(p,q)$ is non-compact. To get an idea why this is so we compute explicitely $SO(1,1)$ which is a closed subgroup of $O(1,1)$, in fact of index two.

$$SO(1,1) = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : \begin{matrix} x^2 - y^2 = 1 \\ x, y \in \mathbb{R} \end{matrix} \right\}$$

The orbit of e_1 under $SO(1,1)$ is



We can parametrize the hyperbola containing e_1 by $t \mapsto \begin{pmatrix} \operatorname{ch} t \\ \operatorname{sh} t \end{pmatrix}$, $t \in \mathbb{R}$

and in fact (exercise)

$$\begin{aligned} \mathbb{R} &\longrightarrow SO(1,1) \\ t &\longmapsto \left\{ \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} : t \in \mathbb{R} \right\} \end{aligned}$$

is a continuous group homomorphism identifying $\mathbb{R}$ topologically with $SO(1,1)^\circ$

the connected component of $SO(1,1)$ containing Id.

Exercise: use $p \geq 1$, $q \geq 1$ to inject $\mathbb{R}$ isomorphically to a closed subgroup of $O(p,q)$.

Example 2.21: (see Iozzi's notes Example 2.24 for details).

Let $\mathcal{H}$ be a complex Hilbert space. One can endow the group $U(\mathcal{H})$ of unitary operators with the strong operator topology; a basis of open sets is given by

$\mathcal{U}(T; u_1, \dots, u_n; \varepsilon)$

$= \{ S \in U(\mathcal{H}) : \| S u_i - T u_i \| < \varepsilon, 1 \leq i \leq n \}$

One verifies that $\mathcal{U}(\mathcal{H})$ is a topological group and: (see lemma 2.2. in Iozzi),

$\mathcal{U}(\mathcal{H})$ is locally compact $\Leftrightarrow$ $\dim \mathcal{H} < +\infty$ in which case it is compact.

2.3. General facts about topological groups.

Recall some facts about connectedness in topological spaces. A topological space is connected if it cannot be written as disjoint union of two proper open subsets. Recall:

- the closure of a connected subset is connected
- the continuous image of a connected set is connected.

Given a top. space X the relation

$$x \sim y \quad \text{iff } \{x, y\} \subset \text{connected subset}$$

is an equivalence relation and its equivalence classes are called the connected components of X. They are the maximal connected subsets.

We have:

<u>Prop. 2.22</u> Let G be a topological group then the following hold:

(i) If $H \leq G$ is a subgroup so is $\overline{H}$.

(ii) If $H \leq G$ is open then it is closed.

(iii) The connected component G° of G containing the neutral element e is a closed normal subgroup

(iv) If G is connected and $U \ni e$ a neighborhood of e then $\bigcup_{n \geq 1} U^n = G$.

(v) If G is connected and $N \lhd G$ is discrete normal, then N is contained in the center $Z(G)$ of G.

<u>Notation</u>: – given subset $A_1, \dots, A_n$ in G

$$A_1 \cdots A_n = \{ a_1 \cdots a_n : a_i \in A_i \}$$

– given $\mathcal{U} \subset G$,

$$\mathcal{U}^n := \{ u_1 \cdots u_n : u_i \in \mathcal{U} ; 1 \le i \le n \}$$

$$\mathcal{U}^{-1} := \{ u^{-1} : u \in \mathcal{U} \}$$

– we say a subset $V \subset G$ is symmetric if $V^{-1} = V$.

<u>Lemma 2.23</u>

(i) If $U \ni e$ is a neighborhood of e there exists $V \ni e$ symmetric open with $V \subset U$.

(ii) Let $\mathcal{U} \ni e$ be a neighborhood of e. Then there is $V \ni e$ open symmetric with $V^2 = V^{-1} \cdot V \subset \mathcal{U}$.

<u>Proof:</u> (i) Let $e \in W \subset \mathcal{U}$ with W open. Then $W^{-1} = i(W) \ni e$ is open and $V := W \cap W^{-1} \ni e$ is open symmetric with $V \subset \mathcal{U}$.

(ii) $m: G \times G \to G$ being continuous at (e,e) there is $W \ni e$ neighborhood of e such that

$$W^2 = m(W \times W) \subset \mathcal{U}.$$

Now (by (i)) take $V \ni e$ open symm. with $V \subset W$. $\boxed{3}$

Proof of Prop 2.22.

(i) Since m is continuous

$$m(\overline{H \times H}) \subset \overline{m(H \times H)} = \overline{H}$$

and thus $m(\overline{H} \times \overline{H}) \subset \overline{H}$.

Since $i : G \to G$ is a homeo we got from $i(H) = H$ that $i(\overline{H}) = \overline{H}$ hence $\overline{H}$ is a subgroup.

(ii) Let $\mathcal{R}$ be a set of representatives of G/H with $\mathcal{R} \ni e$. Then

$$G = H \sqcup \bigsqcup_{r \in \mathcal{R} \setminus \{e\}} r.H$$

since $r.H = L_r(H)$, $r.H$ is open and so is $\bigcup_{r \in \mathcal{R} \setminus \{e\}} r.H$ hence H is closed.

(iii) We have $G^\circ \times G^\circ \ni (e,e)$ is connected hence $m(G^\circ \times G^\circ) \ni e$ is and hence $m(G^\circ \times G^\circ) \subset G^\circ$.

Since i is a homeo and $i(e) = e$ we get $i(G^\circ) = G^\circ$. This shows that G° is a subgroup.

Since $\overline{G^\circ} \supset G^\circ \ni e$ is connected, we have by maximality of G° that $\overline{G^\circ} = G^\circ$, hence is closed.

Let $\forall g \in G$: $\mathrm{int}(g) : G \to G$
$x \mapsto g x g^{-1}$

which is clearly continuous. Since $\mathrm{int}(g)(e) = e$ we get $\mathrm{int}(g)(G^\circ) \subset G^\circ$ hence G° is normal in G.

(iv) Let $V = V^{-1} \ni e$ open with $V \in \mathcal{U}$. Then

$$H := \bigcup_{n \geq 1} V^n \subset \bigcup_{n \geq 1} U^n .$$

One verifies that H is a subgroup; in addition each V^n is open hence H is an open subgroup hence closed. This implies $H = G$ and hence $\bigcup_{n \geq 1} U^n = G$.

(v) Let $N \triangleleft G$ be discrete, normal. Then, $\forall\, n \in N$ the continuous map

$$\begin{aligned} G &\longrightarrow N \\ g &\longmapsto g n g^{-1} \end{aligned}$$

has connected image containing n; since N is discrete, this image is n hence $N \subset Z(G)$. $\boxed{\varepsilon}$