Now we turn to Engel's theorem which is an analogue of Lie's theorem (Thm 4.28) for solvable Lie algebras. However consider:

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \; * \in \mathbb{R} \right\}$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} : * \in \mathbb{R} \right\}$$

which are both nilpotent. While $\mathfrak{n}$ has strict upper triangular form, there is no change of basis that would make $\mathfrak{a}$ upper triangular.

What replaces then Lie's theorem is:

Thm. 4.35 Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ into an $\mathbb{R}$-vector space such that $\rho(X)$ is nilpotent $\forall X \in \mathfrak{g}$. Then there is a basis of V wrt which $\rho(\mathfrak{g})$ is strictly upper triangular.

An important role is played by

<u>Def. 4.36</u> $v \in V$ is a common nullvektor of a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ if

$$\rho(x)v = 0 \quad \forall x \in \mathfrak{g}.$$

We denote by V_0 the subspace of common nullvektors.

Engels thm will follow by recurrence on the dimension of V from:

<u>Thm. 4.37</u> Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation such that $\rho(x)$ is nilpotent $\forall x \in \mathfrak{g}$. Then $V_0 \neq (0)$.

<u>Lemma 4.38</u> Assume $X \in \mathfrak{gl}(V)$ is nilpotent. Then $\mathrm{ad}(X) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is nilpotent.

Proof: Assume $X^{\ell} = 0$ and define the following endomorphisms of $g\ell(V)$:

$$\ell_X : Y \to XY \text{ and } r_X : Y \to YX.$$

Then $ad(X) = \ell_X - r_X$ and ℓ_X, r_X commute. Thus

$$ad(X)^{2\ell} = \sum_{k=0}^{2\ell} \binom{2\ell}{k} \ell_X^k r_X^{2\ell-k}$$

But $r_X^{\ell} = 0$, $\ell_X^{\ell} = 0$ and $\max(k, 2\ell-k) \geq \ell$ $\forall\, 0 \leq k \leq 2\ell$. $\boxed{\xi}$

Proof of Thm 4.37: By induction on $\dim g$.

(1) $g = \mathbb{R} \cdot X$, $\rho(X)$ nilpotent, thus there is $v \neq 0$ with $\rho(X) v = 0$. Thus $V_0 \neq (0)$.

(2) $\dim g \geq 2$. We may assume ρ injective since otherwise, $\dim \operatorname{Ker} \rho \geq 1$ and we are done by recurrence, as $\dim (g/_{\operatorname{Ker}\rho}) < \dim g$.

Via ρ we may thus identify $\mathfrak{g}$ with a Lie subalgebra $\mathfrak{g} < \mathfrak{gl}(V)$ such that every $X \in \mathfrak{g}$ ~~is~~ is a nilpotent endomorphism of V. Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal proper subalgebra. Then by lemma 4.38, $\forall X \in \mathfrak{h}$ $ad(X) : \mathfrak{gl}(V) \circlearrowleft$ is nilpotent, preserves $\mathfrak{g}, \mathfrak{h}$ and hence defines an endomorphism

$$\overline{ad(X)} : \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h}.$$

Hence we get a representation

$$\begin{aligned} \mathfrak{h} &\longrightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) \\ X &\longmapsto \overline{ad(X)} \end{aligned}$$

consisting of nilpotent elements. By recurrence there is $X_0 \in \mathfrak{g}$, $X_0 \notin \mathfrak{h}$ such that its image $\overline{X}_0 \in \mathfrak{g}/\mathfrak{h} \setminus (0)$ is a common null vector for $\mathfrak{h}$. But this means $\forall X \in \mathfrak{h}$:

$$[X, X_0] \in \mathfrak{h}.$$

Thus $\mathbb{R}X_0 + \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$:
for $Y, Z \in \mathfrak{h}$, $\lambda, \mu \in \mathbb{R}$:

$$[\lambda X_0 + Y, \mu X_0 + Z] = \lambda \underset{\mathfrak{h}}{[X_0, Z]} + \mu \underset{\mathfrak{h}}{[Y, X_0]} + \underset{\mathfrak{h}}{[Y, Z]}.$$

By maximality of $\mathfrak{h}$ we have $\mathfrak{g} = \mathbb{R}X_0 + \mathfrak{h}$

and $\mathfrak{h} \triangleleft \mathfrak{g}$.

Now by induction, the space W_0 of common nullvectors for the $\mathfrak{h}$ action on V is $\neq (0)$.

But $\forall w \in W_0$, $\forall X \in \mathfrak{h}$, $Y \in \mathfrak{g}$:

$$XYw = (XY - YX)w + YXw$$

$$= [X, Y]w = 0$$

and hence W_0 is $\mathfrak{g}$-invariant. Since X_0 is nilpotent so is $X_0|_{W_0}$ and there exists $v_0 \in W_0$, $v_0 \neq 0$ with $X_0 v_0 = 0$.

Then v_0 is common nullvector for $\mathfrak{g}$.

$\boxed{\xi}$

<u>Proof of Thm 4.35</u>: Clear by induction on $\dim V$. ∎

The relationship between nilpotent Lie algebras and strict upper triangular matrices is then:

<u>Corollary 4.39</u> $\mathfrak{g}$ is nilpotent $\iff$ $\mathrm{ad}(\mathfrak{g})$ is strict upper triangular.

<u>Proof</u>: ($\Leftarrow$) If there is a basis of $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \right\}$ then $\mathfrak{n}$ and $\mathrm{ad}(\mathfrak{g})$ are nilpotent. In addition $\mathrm{Ker}\,\mathrm{ad} = Z(\mathfrak{g})$, hence by lemma 4.34 (2), $\mathfrak{g}$ is nilpotent.

($\Rightarrow$) If $\mathfrak{g}$ is nilpotent, then in particular by Prop. 4.31 (3), $\mathrm{ad}(X)$ is nilpotent $\forall X \in \mathfrak{g}$, so that the assertion follows from Thm. 4.35. $\boxed{B}$

In analogy to the case of Lie algebras there is a notion of nilpotent group to which we briefly turn.

<u>Def. 4.40.</u> Given a group G and subgroups A, B in G we define:

$[A,B]$ = subgroup of G generated by

$$\{[a,b] : a \in A,\ b \in B\}$$

where we recall that $[a,b] = aba^{-1}b^{-1}$.

<u>Remark 4.41.</u> If $A \triangleleft G$ and $B \triangleleft G$ then $[A,B] \triangleleft G$.

<u>Def. 4.41.</u> A group G is nilpotent if there is a sequence

$$G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_r = (e)$$

with $[G, G_{i-1}] \subset G_i \qquad 1 \leq i \leq r.$

Clearly this condition implies in particular that $G_{i-1} \triangleleft G$, $1 \leq i \leq r$.

and the condition $[G, G_{i-1}] \subset G_i$

is then equivalent to

$$G_{i-1}/G_i \subset Z(G/G_i).$$

As in the case of Lie algebras one defines inductively:

$$C^1(G) := [G, G],$$

and $\quad C^i(G) := [G, C^{i-1}(G)] \quad 2 \leq i.$

<u>Def 4.42.</u> $C^i(G)$ is the descending central series of G.

One has then:

<u>Lemma 4.43</u> G is nilpotent iff there is $r \geq 1$ s.t. $C^r(G) = \{e\}$.

One shows then with the same methods as in the case of solvable groups:

<u>Thm 4.44</u> Let G be a connected Lie group. TFAE:

(1) G is nilpotent

(2) There is a sequence of closed connected subgroups

$$G = G_0 > G_1 > \cdots > G_r = (e)$$

with $[G, G_{i-1}] \subset G_i \quad 1 \leq i \leq r$.

(3) $\mathfrak{g} = \mathrm{Lie}(G)$ is a nilpotent Lie algebra.

4.3. The Killing form and Cartan's criterion for solvability.

The Killing form plays a fundamental role in the study of semisimple Lie algebras via Cartan's criterion for solvability to which we now turn.

From now on $\mathfrak{g}$ will be a $\mathbb{K}$-Lie algebra where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

<u>Def. 4.45</u> The Killing form of a $\mathbb{K}$-Lie algebra is the bilinear form

$$K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K}$$

defined by $K_{\mathfrak{g}}(X, Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y))$.

Its importance stems from the following invariance property:

Prop. 4.46. $K_{\mathfrak{g}}(ad(Z)X, Y) + K_{\mathfrak{g}}(X, ad(Z)Y) = 0$

$\forall X, Y, Z \in \mathfrak{g}$.

Proof:

$$tr\left(ad([Z,X])\,ad(Y)\right) + tr\left(ad(X)\,ad[Z,Y]\right)$$

$$= tr\left([ad Z, ad X]\,ad(Y)\right) + tr\left(ad(X)[ad(Z), ad(Y)]\right)$$

$$= tr\left(ad Z\, ad(X)\, ad(Y)\right) - tr\left(ad(X)\,ad(Z)\,ad(Y)\right)$$

$$+ tr\left(ad(X)\,ad(Z)\,ad(Y)\right) - tr\left(ad(X)\,ad(Y)\,ad(Z)\right)$$

$$= tr\left(ad(Z)\,\underbrace{ad(X)\,ad(Y)}\right) - tr\left(ad(X)\,ad(Y)\,ad(Z)\right)$$

$= 0$ $\boxed{3}$

Exercise 4.47 Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show:

$$K_{\mathfrak{g}}(Ad(g)X, Ad(g)Y) = K_{\mathfrak{g}}(X, Y)$$

$\forall X, Y \in \mathfrak{g}$, $\forall g \in G$.