<u>Hint</u>: Compute the derivative of

$$\phi(t) = K_{\mathfrak{g}}\left(\mathrm{Ad}(\exp tZ)X, \mathrm{Ad}(\exp tZ)Y\right).$$

The objective of this section is to show:

<u>Cor. 4.48</u> (Cartan criterion). A $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is solvable $\Leftrightarrow K_{\mathfrak{g}}\big|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}} = 0$.

<u>Exercise 4.49</u>

(1) Let $\mathfrak{g}$ be an $\mathbb{R}$-Lie algebra and identify $\mathfrak{g}$ with its image in $\mathfrak{g}_{\mathbb{C}}$. Then $K_{\mathfrak{g}} = K_{\mathfrak{g}_{\mathbb{C}}}\big|_{\mathfrak{g}\times\mathfrak{g}}$.

(2) $K_{\mathfrak{g}}\big|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}} = 0 \Leftrightarrow K_{\mathfrak{g}_{\mathbb{C}}}\big|_{\mathfrak{g}_{\mathbb{C}}^{(1)}\times\mathfrak{g}_{\mathbb{C}}^{(1)}} = 0$.

Hint: use that $(\mathfrak{g}_{\mathbb{C}})^{(1)} = \mathfrak{g}^{(1)} + i\cdot\mathfrak{g}^{(1)}$.

Cartan's criterion will follow from:

<u>Thm 4.50</u> Let V be a complex vector space and $\mathfrak{g} < \mathfrak{gl}(V)$ a Lie subalgebra. If $\mathrm{tr}(XY) = 0$ $\forall X, Y \in \mathfrak{g}$ then $\mathfrak{g}^{(1)}$ can be made strictly upper triangular.

In particular $\mathfrak{g}^{(1)}$ is nilpotent and $\mathfrak{g}$ is solvable.

First we show how the theorem implies the corollary.

<u>Thm 4.50 $\Rightarrow$ Cor. 4.48</u>

(1) $\mathbb{K} = \mathbb{R}$: by Exercise 4.49 we have

$$K_{\mathfrak{g}}\big|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = 0 \iff K_{\mathfrak{g}_{\mathbb{C}}}\big|_{\mathfrak{g}_{\mathbb{C}}^{(1)} \times \mathfrak{g}_{\mathbb{C}}^{(1)}} = 0$$

$\iff$ (Cor. 4.48 for $\mathbb{K} = \mathbb{C}$) $\mathfrak{g}_{\mathbb{C}}$ is solvable

$\iff$ (Exercise) $\mathfrak{g}$ is solvable.

It suffices therefore to show Thm 4.50 $\Rightarrow$ Cor. 4.48. for $\mathbb{K} = \mathbb{C}$.

(2) Assuming Thm 4.50 we show Cor. 4.48 for $\mathbb{K} = \mathbb{C}$.

($\Longrightarrow$) If $\mathfrak{g}$ is solvable ~~and~~ ~~nilpotent~~ then by Lie's theorem $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ can be made upper triangular.

But then $ad(\mathfrak{g}^{(1)}) = (ad(\mathfrak{g}))^{(1)}$ is strictly upper triangular hence $tr(ad(X)ad(Y)) = 0$ $\forall X, Y \in \mathfrak{g}^{(1)}$.

($\Longleftarrow$) If $tr(ad(X)ad(Y)) = 0$ $\forall X, Y \in \mathfrak{g}^{(1)}$ then applying Thm 4.50 to $ad(\mathfrak{g}^{(1)})$ we get that $ad(\mathfrak{g}^{(2)})$ can be made strictly upper triangular; this implies ~~that $\mathfrak{g}^{(2)}$ is nilpotent hence $\mathfrak{g}$~~ that $\mathfrak{g}^{(2)}$ is solvable (even nilpotent) hence $\mathfrak{g}$ is solvable since $\mathfrak{g}/\mathfrak{g}^{(2)}$ is obviously solvable. $\boxed{R}$

The proof of Thm. 4.50 will require the use of the Jordan decomposition which we now recall.

<u>Prop. 4.51</u>

Let $A \in \mathrm{End}(V)$, V a $\mathbb{C}$-vector space.

Then (1) $A = A_s + A_n$ where A_s is diagonalizable, A_n nilpotent, and

(2) $[A_s, A_n] = 0$.

Moreover A_s, A_n are uniquely determined by (1) and (2).

(3) There exist polynomials P_s, P_n without constant term such that

$$A_s = P_s(A), \quad A_n = P_n(A).$$

<u>Lemma 4.52</u> Let $X \in gl(V)$ and $X = X_s + X_n$ its Jordan decomposition. Then $ad(X) = ad(X_s) + ad(X_n)$ is the Jordan decomposition of $ad(X)$.

<u>Proof of lemma 4.52</u>

We already know from lemma 4.38 that $ad(X_n)$ is nilpotent; in addition,

$$[ad(X_s), ad(X_n)] = ad([X_s, X_n]) = 0.$$

Let $e_1, \ldots, e_n$ be a basis of V consisting of eigenvectors of X_s with corr. eigenvalues $\lambda_1, \ldots, \lambda_n$. Then if

$$X_s = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ and } E_{ij} = \begin{array}{c c} & \overbrace{}^{j} \\ i & \begin{pmatrix} & & \vdots & \\ \cdots & \cdots & 1 & \\ & & & \end{pmatrix} \end{array}$$

$$[X_s, E_{ij}] = (\lambda_i - \lambda_j) E_{ij}.$$

which shows that $ad(X_s)$ is diagonalizable and shows the lemma. $\boxed{\beta}$

Lemma 4.53 Let W be a $\mathbb{C}$-vector space and $T \in \mathrm{End}(W)$ diagonalizable with $W = \bigoplus_{i=1}^{r} W_{\lambda_i}$ where $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of T and W_{λ_i} the corr. eigenspaces. Let $\overline{T} \in \mathrm{End}(W)$ with $\overline{T}\big|_{W_{\lambda_i}} = \overline{\lambda_i} \cdot \mathrm{Id}_{W_{\lambda_i}}$.

Then there is a polynomial Q with

$$\overline{T} = Q(T).$$

Proof: Define the interpolation polynomial

$$Q(x) = \sum_{j=1}^{r} \overline{\lambda}_j \prod_{i \neq j} \frac{(x - \lambda_i)}{(\lambda_j - \lambda_i)}$$

then $Q(\lambda_i) = \overline{\lambda_i}$ and hence

$$Q(T) = \overline{T}. \qquad \blacksquare$$

Proof of Thm 4-50.

We show that every $X \in \mathfrak{g}^{(1)}$ is nilpotent and conclude by Engel's theorem.

Let $X = X_s + X_n$ be the Jordan decomposition of $X \in \mathfrak{g}^{(1)} \subset \mathfrak{g} \subset \mathfrak{gl}(V)$ and take a basis of V wrt the matrix of $X_s = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

We will show that

$$\sum_{i=1}^{n} |\lambda_i|^2 = tr(X_s \overline{X}_s) = 0$$

hence concluding $X_s = 0$ and X is nilpotent.

Now $tr(X_s \overline{X}_s) = tr((X - X_n)\overline{X}_s)$

$$= tr(X\overline{X}_s) - tr(X_n \overline{X}_s).$$

We will show separately that:

(1) $tr(X_n \overline{X}_s) = 0$

(2) $tr(X \overline{X}_s) = 0$.

(1) By lemma 4.53 there is a polynomial Q such that $\overline{X}_s = Q(X_s)$; hence X_n and $\overline{X}_s$ commute, and from $X_n^\ell = 0$ for some $\ell \geq 1$ we conclude

$$(X_n \overline{X}_s)^\ell = X_n^\ell \overline{X}_s^\ell = 0$$

hence $tr(X_n \overline{X}_s) = 0$.

(2) Since $X \in g^{(1)}$, $X = \sum_{i=1}^{k} [X_i, Y_i]$ for some $X_i, Y_i \in g$.

Then:

$$tr(X \overline{X}_s) = tr\left(\sum_{i=1}^{k} [X_i, Y_i] \overline{X}_s\right)$$

$$= tr\left(\sum_{i=1}^{k} X_i Y_i \overline{X}_s - Y_i X_i \overline{X}_s\right)$$

$$= tr\left(\sum_{i=1}^{k} Y_i \overline{X}_s X_i - \overline{X}_s Y_i X_i\right)$$

$$= tr\left(\sum_{i=1}^{n} [Y_i, \bar{X}_s] X_i \right)$$

$$= - \sum_{i=1}^{n} tr\left(ad(\bar{X}_s)(Y_i) \cdot X_i \right).$$

We know that $tr(A \cdot B) = 0 \quad \forall A, B \in g$.

Thus it suffices to show that $ad(\bar{X}_s) Y_i \in g$.

Observe that $\overline{ad(X_s)} = ad(\bar{X}_s)$

and thus there is a polynomial Q (lemma 4.53)

with $ad(\bar{X}_s) = Q(ad(X_s))$; in

addition since (lemma 4.52)

$$ad(X) = ad(X_s) + ad(X_n)$$

is the Jordan decomposition of $ad(X)$

there is a polynomial P with $ad(X_s) = P(ad(X))$.

Thus $ad(\bar{X}_s) = Q(P(ad(X)))$

and hence $ad(\bar{X}_s)\, g \subset g$ which concludes

the proof. ☒

4.4. Semisimple Lie algebras and Lie groups.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Def. 4.54 (1) $\mathfrak{g}$ is simple if

(a) $\mathfrak{g}$ is non abelian.

(b) if $\mathfrak{n} \trianglelefteq \mathfrak{g}$ then either $\mathfrak{n} = (0)$ or $\mathfrak{n} = \mathfrak{g}$.

(2) $\mathfrak{g}$ is semisimple if there are simple ideals $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ in $\mathfrak{g}$ such that as Lie algebra:

$$\mathfrak{g} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r .$$

This means that if $X = \sum_{i=1}^{r} X_i$,

$$Y = \sum_{i=1}^{r} Y_i$$

with $X_i, Y_i \in \mathfrak{n}_i$ then:

$$[X, Y] = \sum_{i=1}^{r} [X_i, Y_i] .$$

(3) A connected Lie group G is simple (resp. semisimple) if its Lie algebra is.

$\geq$ $SL(2n, \mathbb{R})$, $n \geq 1$ is a simple Lie group but it is not simple as an abstract group since $Z(SL(2n, \mathbb{R})) = \{\pm Id\}$.

The fundamental characterisation of semisimplicity is:

<u>Thm 4.55</u> $\mathfrak{g}$ is semisimple iff $K_{\mathfrak{g}}$ is non-degenerate.

We will need two lemmas:

<u>Lemma 4.56</u> Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{z} \trianglelefteq \mathfrak{g}$ an ideal. Then

$$\mathfrak{z}^{\perp} = \{ X \in \mathfrak{g} : K_{\mathfrak{g}}(X, Y) = 0 \ \forall Y \in \mathfrak{z} \}$$

is an ideal as well.

<u>Proof</u>: For $Z \in \mathfrak{g}$, $X \in \mathfrak{h}^{\perp}$, $Y \in \mathfrak{h}$:

$$K_{\mathfrak{g}}(\mathrm{ad}(Z)(X), Y) = -K_{\mathfrak{g}}(X, \mathrm{ad}(Z)Y) = 0.$$

∎

Next, if $\mathfrak{h} < \mathfrak{g}$ is just a subalgebra of $\mathfrak{g}$, the restriction of $K_{\mathfrak{g}}$ to $\mathfrak{h} \times \mathfrak{h}$ does not in general coincide with $K_{\mathfrak{h}}$ since for $X, Y \in \mathfrak{h}$ the trace of $\mathrm{ad}(X)\mathrm{ad}(Y)$ is compute wrt $\mathfrak{g}$ and in the second case wrt the restriction to $\mathfrak{h}$. However:

<u>Lemma 4.57</u>: If $\mathfrak{h} \triangleleft \mathfrak{g}$ then $K_{\mathfrak{h}} = K_{\mathfrak{g}}|_{\mathfrak{h} \times \mathfrak{h}}$.

<u>Proof</u>: Let $W \oplus \mathfrak{h} = \mathfrak{g}$ be a vector space complement; then $\forall\, X, Y \in \mathfrak{h}$, $\mathrm{ad}_{\mathfrak{g}}(X)\mathrm{ad}_{\mathfrak{g}}(Y)$ sends $\mathfrak{g}$ into $\mathfrak{h}$ and

and $\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(Y)\big|_{\mathfrak{z}} = \mathrm{ad}_{\mathfrak{z}}(X)\,\mathrm{ad}_{\mathfrak{z}}(Y)$

so that the matrix of $\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(Y)$

wrt any basis adapted to $W \oplus \mathfrak{z} = \mathfrak{g}$

has the form
$$\begin{pmatrix} 0 & * \\ * & \mathrm{ad}_{\mathfrak{z}}(X)\,\mathrm{ad}_{\mathfrak{z}}(Y) \end{pmatrix}$$

which proves the lemma.

<u>Proof of Thm 4.55</u>

($\Longrightarrow$) Assume $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with

$\mathfrak{g}_i$ simple. Then $\forall\ X = \sum_{i=1}^{r} X_i$

$$\mathrm{ad}_{\mathfrak{g}}(X) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{g}_1}(X_1) & & \\ & \ddots & \\ & & \mathrm{ad}_{\mathfrak{g}_r}(X_r) \end{pmatrix}$$

hence $K_g(X,Y) = \sum_{i=1}^{r} K_{g_i}(X_i, Y_i)$.

If now $X \in g^\perp$ this implies $X_i \in g_i^\perp$

where $g_i^\perp = \{Z \in g_i : K_g(Z, A) = 0 \ \forall A \in g_i\}$

Now by lemma 4.56: $g_i^\perp$ is an ideal

in g_i and thus either $g_i = (0)$ or

$g_i^\perp = g_i$. But then $K_{g_i} = 0$ and

Cartan's criterion implies that g_i is

solvable. But then $g_i^{(1)} \neq g_i$ and

g_i being simple this implies $g_i^{(1)} = (0)$

and g_i is abelian: a contradiction with

the definition of simplicity.

Thus $g_i^\perp = (0)$ $\quad \forall\ 1 \leq i \leq r$ which

implies $g^\perp = (0)$ and K_g is non-deg.

∎

($\Leftarrow$) Let's assume $K_{\mathfrak{g}}$ is non-degenerate.

We proceed in 3 steps:

(1) $\mathfrak{g}$ has no nonzero abelian ideal.

Let $\mathfrak{a} \triangleleft \mathfrak{g}$ be an abelian ideal and let $W \oplus \mathfrak{a} = \mathfrak{g}$ be a vector space complement. Then $\forall\, X \in \mathfrak{a}$

$$\operatorname{ad}(X) = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$$

and $\forall\, Y \in \mathfrak{g}$:

$$\operatorname{ad}(Y) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

and hence $K_{\mathfrak{g}}(X, Y) = 0$; thus

$$\mathfrak{a} \subset \mathfrak{g}^{\perp} = (0).$$

(2) If $\mathfrak{h} \triangleleft \mathfrak{g}$ then $\mathfrak{h}^{\perp} \triangleleft \mathfrak{g}$ and

$$\mathfrak{h} \cap \mathfrak{h}^{\perp} = (0).$$

We know from lemma 4.56 that $\mathfrak{z} \cap \mathfrak{z}^\perp$ is an ideal. Moreover: $\forall Z \in \mathfrak{g}$, $\forall X, Y \in \mathfrak{z} \cap \mathfrak{z}^\perp$ we have

$$K_{\mathfrak{g}}([X,Y], Z) = -K_{\mathfrak{g}}(\underset{\mathfrak{z}}{\underset{\cap}{Y}}, \underset{\mathfrak{z}^\perp}{\underset{\cap}{[X,Z]}}) = 0$$

and since $K_{\mathfrak{g}}$ is non-degenerate this implies $[X,Y]=0$ $\forall X, Y \in \mathfrak{z} \cap \mathfrak{z}^\perp$ and $\mathfrak{z} \cap \mathfrak{z}^\perp$ is an abelian ideal hence by (1) $\mathfrak{z} \cap \mathfrak{z}^\perp = (0)$.

(3) Since $K_{\mathfrak{g}}$ is non-degenerate we have $\dim \mathfrak{z}^\perp = \dim \mathfrak{g} - \dim \mathfrak{z}$ and hence as vector spaces: $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ but $\mathfrak{z}, \mathfrak{z}^\perp$ being both ideal this is a Lie algebra direct sum. We have thus $K_{\mathfrak{g}} = K_{\mathfrak{g}}|_{\mathfrak{z}\times\mathfrak{z}} \oplus K_{\mathfrak{g}}|_{\mathfrak{z}^\perp\times\mathfrak{z}^\perp}$

which by lemma 5.57

$$= K_{\mathfrak{z}} \oplus K_{\mathfrak{z}^\perp}.$$

Thus $K_{\mathfrak{z}}$ and $K_{\mathfrak{z}^\perp}$ are non-degenerate.

We conclude by recurrence on dim $\mathfrak{g}$.

$\boxtimes$

Next we establish a powerful way to produce families of semisimple algebras.

<u>Theorem 4.58</u>. Let V be a $\mathbb{K}$-vector space endowed with an inner product $\langle\, ,\, \rangle$.
If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a $\mathbb{K}$-subalgebra that is self-adjoint ~~then $\mathbb{K}$~~ and $Z(\mathfrak{g}) = 0$ then $K_{\mathfrak{g}}$ is non-degenerate and $\mathfrak{g}$ is semisimple.