More precisely: let for $A \in gl(V)$, $A^* \in gl(V)$ be defined by:

$$\langle Av, w \rangle = \langle v, A^* w \rangle \quad \forall v, w \in V.$$

Then g self-adjoint means that $\forall A \in g$, $A^* \in g$.

Proof of Thm.

In the sequel if $T \in End(W)$ we will denote by $tr_W(T)$ the trace of T as endomorphism of W.

In the proof we assume $\mathbb{K} = \mathbb{R}$; $\mathbb{K} = \mathbb{C}$ is analogous.

With $\langle \, , \rangle$ and $\underset{gl(V) \to gl(V)}{A \mapsto A^*}$ as above we define a scalar product on $gl(V)$:

$$\langle\langle A, B \rangle\rangle := tr_V (A \cdot B^*), \quad A, B \in gl(V).$$

Let's just verify ~~...~~ positivity:

$$= tr_V\,(AB^*X) - tr_V\,(AXB^*)$$

$$= tr_V\left(A\left((X^*B)^* - (BX^*)^*\right)\right)$$

$$= tr_V\,(A\,[X^*, B]^*)$$

$$= \langle\langle A, \mathrm{ad}_{\mathfrak{g}}(X^*)\,B \rangle\rangle$$

which shows the claim.

Thus $\mathrm{ad}(\mathfrak{g})$ is a Θ-invariant subalg. of $\mathfrak{gl}(\mathfrak{g})$. Let then $E_1, \dots, E_m$ be an orthonormal basis of $\mathfrak{g}$ wrt $\langle\langle\ \rangle\rangle$.

Then if $A \in \mathfrak{gl}(m, \mathbb{R})$, $\Theta(A) = {}^tA$.

Hence $\forall\, X \in \mathfrak{g}$:

$$tr_{\mathfrak{g}}\,(\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(X^*)) = tr_{\mathfrak{g}}\,(\underbrace{\mathrm{ad}_{\mathfrak{g}}(X)}_{A}\ \Theta\,(\mathrm{ad}_{\mathfrak{g}}(X)))$$

$= tr\,({}^tA\,A) \geq 0$ with equality $\Leftrightarrow A = 0$

that is $\mathrm{ad}_{\mathfrak{g}}(X) = 0$; hence $X \in Z(\mathfrak{g}) = \{0\}$.

$\boxed{\Sigma}$

Let $e_1, \dots, e_n$ be an orthonormal basis of V

then if $A = (a_{ij})$, $A^* = {}^t(a_{ij})$

and $tr_V(AA^*) = \sum_{i,j=1}^{n} a_{ij}^2 \geqslant 0$

with equality $\iff A = 0$.

Now we restrict this scalar product to $\mathfrak{g}$; let $\theta : \mathfrak{gl}(\mathfrak{g}) \longrightarrow \mathfrak{gl}(\mathfrak{g})$ be the adjoint, that is ~~if $T, S \in \mathfrak{gl}(\mathfrak{g})$~~ if $T \in \mathfrak{gl}(\mathfrak{g})$, $A, B \in \mathfrak{g}$ then:

$$\langle\langle T(A), B \rangle\rangle = \langle\langle A, \theta(T)(B) \rangle\rangle$$

$$\forall A, B \in \mathfrak{g}.$$

We claim: $ad_{\mathfrak{g}}(X^*) = \theta(ad_{\mathfrak{g}}(X))$ $\forall X \in \mathfrak{g}$.

Indeed $\forall A, B \in \mathfrak{g}$:

$\langle\langle Ad_{\mathfrak{g}}(X) A, B \rangle\rangle = tr_V([X, A] B^*)$

$= tr_V(XAB^*) - tr_V(AXB^*) =$

Thm 4.58 now easily leads to family's of examples of semisimple Lie algebras.

Examples 4.59.

(1) $\mathfrak{sl}(n,\mathbb{R}) \subset \mathfrak{gl}(n,\mathbb{R})$ is invariant under $A \mapsto {}^tA$ and $Z(\mathfrak{sl}(n,\mathbb{R})) = (0)$.

(2) $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$ is invariant under $A \longmapsto {}^t\bar{A}$ and $Z(\mathfrak{sl}(n,\mathbb{C})) = 0$.

(3) $p+q=n$, $\mathfrak{o}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{R}) :$

$${}^tX J_{p,q} + J_{p,q} X = 0\}$$

where $J_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$ is invariant under $X \mapsto {}^tX$: indeed from ${}^tX J_{p,q} + J_{p,q} X = 0$ we obtain by multiplying on the left and right with $J_{p,q}$: $J_{p,q}\,{}^tX + X J_{p,q} = 0$ since $J_{p,q}^2 = \mathrm{Id}_n$.

One verifies that $Z(\mathfrak{o}(p,q)) = (0)$.

(4) $sp(2n, \mathbb{R}) = \{ X \in gl(n, \mathbb{R}) : {}^tXJ + JX = 0 \}$

with $J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$, is invariant under

$X \mapsto {}^tX$, as a verification analogous

to the one in (3) shows. Also $Z(sp(2n, \mathbb{R})) = 0$!

It will turn out to be important to have

a description of all ideals of a semisimple

Lie algebra.

<u>Prop. 4.60</u> Let $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ be a direct

sum of simple ideals. Then any ideal $\mathfrak{z}$ in

$\mathfrak{g}$ is of the form $\bigoplus_{i \in J} \mathfrak{g}_i$ for some $J \subset I$.

<u>Proof:</u> Let $J \subset I$ be minimal with $\mathfrak{z} \subset \bigoplus_{j \in J} \mathfrak{g}_j$.

Since $\mathfrak{g}_j \triangleleft \mathfrak{g}$, $[\mathfrak{z}, \mathfrak{g}_j] \subset \mathfrak{g}_j$ and since

$\mathfrak{g}_j$ is simple either $[\mathfrak{z}, \mathfrak{g}_j] = \mathfrak{g}_j$ implying

$\mathfrak{g}_j = [\mathfrak{h}, \mathfrak{g}_j] \subset \mathfrak{h}$ since $\mathfrak{J}$ is an ideal,

or $[\mathfrak{h}, \mathfrak{g}_j] = 0$. Let's exclude the latter

case; let $X \in \mathfrak{h}$ and write $X = \sum_{\ell \in J} X_\ell$.

Then $[\mathfrak{h}, \mathfrak{g}_j] = 0$ implies $\forall\, Y_j \in \mathfrak{g}_j$:

$0 = [X, Y_j] = [X_j, Y_j]$ which implies

$X_j \in Z(\mathfrak{g}_j) = (0)$. This would hold

for any $X \in \mathfrak{h}$ and contradict the mini-

mality of J. Thus we are in the first

case, hence $\mathfrak{g}_j \subset \mathfrak{h}$ $\forall j \in J$ which

implies $\mathfrak{h} = \bigoplus_{j \in J} \mathfrak{g}_j$. $\boxed{z}$

Now we are going to put everything together

and obtain a structure theorem for

semisimple connected groups.

Let G be connected semisimple; then its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ is a direct sum of simple ideals. In general one can not expect G to be a direct product of simple Lie groups.

<u>Example 4.61</u> Let $Z = \{(\pm Id)\}$ be the center of $SL(2,\mathbb{R})$ and let

$$D = \{(Id, Id), (-Id, -Id)\} \subset Z^2.$$

Define $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) / D$.

Let $G_1 = pr(SL(2,\mathbb{R}) \times \{Id\})$

$G_2 = pr(\{Id\} \times SL(2,\mathbb{R}))$

then $Lie(G_1) = \mathfrak{sl}(2,\mathbb{R}) \oplus (0)$

$Lie(G_2) = (0) \oplus \mathfrak{sl}(2,\mathbb{R})$

but $G_1 \cap G_2 = Z \times Z / D$ which has order two.

In fact this example reflects the general case:

<u>Thm 4.62</u> Let G be a connected semisimple Lie group and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ the decomposition of its Lie algebra into simple ideals. Then there are closed connected normal subgroups $G_i \triangleleft G$ with $\mathrm{Lie}(G_i) = \mathfrak{g}_i$ such that the product map

$$m : G_1 \times \cdots \times G_r \longrightarrow G$$
$$(g_1, \ldots, g_r) \longmapsto g_1 \cdots g_r$$

is a surjective homomorphism with

$$\mathrm{Ker}\, m \subset \prod_{i=1}^{r} Z(G_i).$$

<u>Remark 4.63</u> Since $\mathrm{Lie}\, Z(G_i) = Z(\mathfrak{g}_i) = 0$ the subgroup $Z(G_i) < G_i$ is discrete; for the same reason $Z(G)$ is a discrete subgroup of G.

<u>Lemma 4.64.</u> Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n}_1, \mathfrak{n}_2$ ideals in $\mathfrak{g}$ <u>with $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$</u>. Let $N_i = \langle \exp \mathfrak{n}_i \rangle$. Then N_i are closed connected and $[N_1, N_2] = e$.

<u>Proof:</u> $\mathrm{ad}(X_1)(X_2) = 0 \quad \forall\, X_i \in \mathfrak{n}_i$.

So $\underbrace{\mathrm{Exp}\,(\mathrm{ad}(X_1))}_{}(X_2) = X_2$

$\mathrm{Ad}(\exp(X_1))(X_2)$

But then: $\underbrace{\exp\big(\mathrm{Ad}(\exp(X_1))(X_2)\big)}_{\mathrm{Int}(\exp(X_1))(\exp X_2)} = \exp X_2$

that implies $\exp \mathfrak{n}_1$ commutes with $\exp \mathfrak{n}_2$, so does $\langle \exp \mathfrak{n}_1 \rangle$ and $\langle \exp \mathfrak{n}_2 \rangle$ and hence N_1 and N_2 commute.

$\square$

<u>Proof of Thm. 4.62:</u> We may assume that $r \geq 2$. Then by Lemma 4.64, $G_i := \langle \exp \mathfrak{g}_i \rangle$ is closed connected normal in G and

$$[G_i, G_j] = (e) \qquad i \neq j.$$

But then $[\mathrm{Lie}(G_i), \mathrm{Lie}(G_j)] = 0$, $i \neq j$.

Now $\mathfrak{g}_i \subset \mathrm{Lie}(G_i) \lhd \mathfrak{g}$ and if for some i there is not equality, then by Prop. 4.60 there is $j \neq i$ such that

$$\mathrm{Lie}(G_i) \supset \mathfrak{g}_j .$$

But $\mathrm{Lie}(G_j) \supset \mathfrak{g}_j$ and this would imply $[\mathfrak{g}_j, \mathfrak{g}_j] = 0$ contradiction since $\mathfrak{g}_j$ is simple. Thus:

$$\mathrm{Lie}(G_i) = \mathfrak{g}_i \quad \forall \quad 1 \leq i \leq r.$$

Since $\forall\ i \neq j$, $[G_i, G_j] = e$, the map

$$m : G_1 \times \cdots \times G_r \longrightarrow G$$
$$(g_1, \ldots, g_r) \longmapsto g_1 \cdots g_r$$

is a homomorphism.

Its derivative is then

$$D_e m(X_1, \ldots, X_r) = \sum_{i=1}^{r} D_e m(0, \ldots, 0, X_i, 0, \ldots, 0)$$
$$= \sum_{i=1}^{r} X_i$$

and hence $D_e m$ implements the vector space isomorphism

$$\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r \longrightarrow \mathfrak{g}$$
$$(X_1, \ldots, X_r) \longmapsto \sum_{i=1}^{r} X_i .$$

Thus m near $(e, \ldots, e)$ is a local diffeomorphism which implies ~~on one side that~~ $m(G_1 \times \cdots \times G_r)$ is open in G hence since G is connected

$$m(G_1 \times \cdots \times G_r) = G .$$

Let now $(g_1, \ldots, g_r) \in \mathrm{Ker}\, m$. Then

$$g_1 \cdots g_r = (e)$$

But then: $g_i^{-1} = g_1 \cdots g_{i-1} g_{i+1} \cdots g_r$

that is $g_i^{-1} \in G_i \cap m(\prod_{j \neq i} G_j)$

Since

G_i and $m(\prod_{j \neq i} G_j)$ commute

we get $g_i \in Z(G_i)$ and thus

$$\mathrm{Ker}\, m \subset \prod_{i=1}^{r} Z(G_i). \qquad \boxed{\S}$$