is an equivalence relation and its equivalence classes are called the connected components of X. They are the maximal connected subsets.

We have:

<u>Prop. 2.22</u> Let G be a topological group then the following hold:

(i) If $H \leqslant G$ is a subgroup so is $\overline{H}$.

(ii) If $H \leqslant G$ is open then it is closed.

(iii) The connected component G° of G containing the neutral element e is a closed normal subgroup

(iv) If G is connected and $U \ni e$ a neighborhood of e then $\bigcup_{n \geq 1} U^n = G$.

(v) If G is connected and $N \lhd G$ is discrete normal, then N is contained in the center $Z(G)$ of G.

<u>Notation</u> :- given subset $A_1, \dots, A_n$ in G

$$A_1 \cdots A_n = \{ a_1 \cdots a_n : a_i \in A_i \}$$

- given $\mathcal{U} \subset G$,

$$\mathcal{U}^n := \{ u_1 \cdots u_n : u_i \in \mathcal{U} ; 1 \le i \le n \}$$

$$\mathcal{U}^{-1} := \{ u^{-1} : u \in \mathcal{U} \}$$

- we say a subset $V \subset G$ is symmetric if $V^{-1} = V$.

<u>Lemma 2.23</u>

(i) If $U \ni e$ is a neighborhood of e there exists $V \ni e$ symmetric open with $V \subset U$.

(ii) Let $\mathcal{U} \ni e$ be a neighborhood of e. Then there is $V \ni e$ open symmetric with $V^2 = V^{-1} \cdot V \subset \mathcal{U}$.

<u>Proof:</u> (i) Let $e \in W \subset \mathcal{U}$ with W open. Then $W^{-1} = i(W) \ni e$ is open and $V := W \cap W^{-1} \ni e$ is open symmetric with $V \subset \mathcal{U}$.

(ii) $m : G \times G \to G$ being continuous at (e, e) there is $W \ni e$ neighborhood of e such that

$$W^2 = m(W \times W) \subset \mathcal{U}.$$

Now (by (i)) take $V \ni e$ open symm. with $V \subset W$. $\boxed{\Xi}$

Proof of Prop 2.22.

(i) Since m is continuous

$$m(\overline{H \times H}) \subset \overline{m(H \times H)} = \overline{H}$$

and thus $m(\overline{H} \times \overline{H}) \subset \overline{H}$.

Since $i: G \to G$ is a homeo we got from $i(H) = H$ that $i(\overline{H}) = \overline{H}$ hence $\overline{H}$ is a subgroup.

(ii) Let $\mathcal{R}$ be a set of representatives of G/H with $\mathcal{R} \ni e$. Then

$$G = H \sqcup \bigsqcup_{r \in \mathcal{R} \setminus \{e\}} r.H$$

since $r.H = L_r(H)$, $r.H$ is open and so is $\bigcup_{r \in \mathcal{R} \setminus \{e\}} r.H$ hence H is closed.

(iii) We have $G^\circ \times G^\circ \ni (e,e)$ is connected hence $m(G^\circ \times G^\circ) \ni e$ is and hence $m(G^\circ \times G^\circ) \subset G^\circ$.

Since i is a homeo and $i(e) = e$ we get $i(G^\circ) = G^\circ$. This shows that G° is a subgroup.

Since $\overline{G^\circ} \supset G^\circ \ni e$ is connected, we have by maximality of G° that $\overline{G^\circ} = G^\circ$, hence is closed.

Let $\forall g \in G$: $\mathrm{int}(g) : G \to G$
$x \mapsto g x g^{-1}$

which is clearly continuous. Since $\mathrm{int}(g)(e) = e$ we get $\mathrm{int}(g)(G^\circ) \subset G^\circ$ hence G° is normal in G.

(iv) Let $V = V^{-1} \ni e$ open with $V \in \mathcal{U}$. Then

$$H := \bigcup_{n \geq 1} V^n \subset \bigcup_{n \geq 1} U^n .$$

One verifies that H is a subgroup; in addition each V^n is open hence H is an open subgroup hence closed. This implies $H = G$ and hence $\bigcup_{n \geq 1} U^n = G$.

(v) Let $N \triangleleft G$ be discrete, normal. Then, $\forall\, n \in N$ the continuous map

$$\begin{aligned} G &\longrightarrow N \\ g &\longmapsto g n g^{-1} \end{aligned}$$

has connected image containing n; since N is discrete, this image is n hence $N \subset Z(G)$. $\boxed{\text{}}$

<u>Remark 2.24</u> For a topological space X, $\pi_0(X)$ denotes the set of connected components; According to Prop. 2.22 (3) we can identify $\pi_0(G)$ with G/G_0 when G is a top. group, and hence $\pi_0(G)$ acquires a group structure. We urge the reader to "compute" $\pi_0(G)$ for the examples of top. groups we listed. Some are difficult to answer, for instance if $G = \mathrm{Homeo}(M)$ with M a compact manifold. We know that if $M = T^2$ the 2-torus then $\pi_0(\mathrm{Homeo}(T^2)) \simeq GL(2, \mathbb{Z})$ and if $S_g :=$  compact orientable surface of genus $g \geq 2$, $\pi_0(\mathrm{Homeo}(S_g))$ is the mapping class group.

Let us explicit the relation of Prop. 2.22 (ii) with covering theory of topological groups.

Recall that if X is path connected, locally path connected and semi locally simply connected, then a universal covering of the pointed space $(X, *)$ always exists. We leave the following as an

<u>Exercise:</u> if H is a top. group and $p : G \to H$ a covering <u>space</u>, then $\forall\, e^* \in p^{-1}(e)$ there exists a unique topological group structure on G with neutral element e^* such that $p : G \longrightarrow H$ is a homomorphism.

Assume now that the top. group H satisfies the hypothesis of covering theory and let

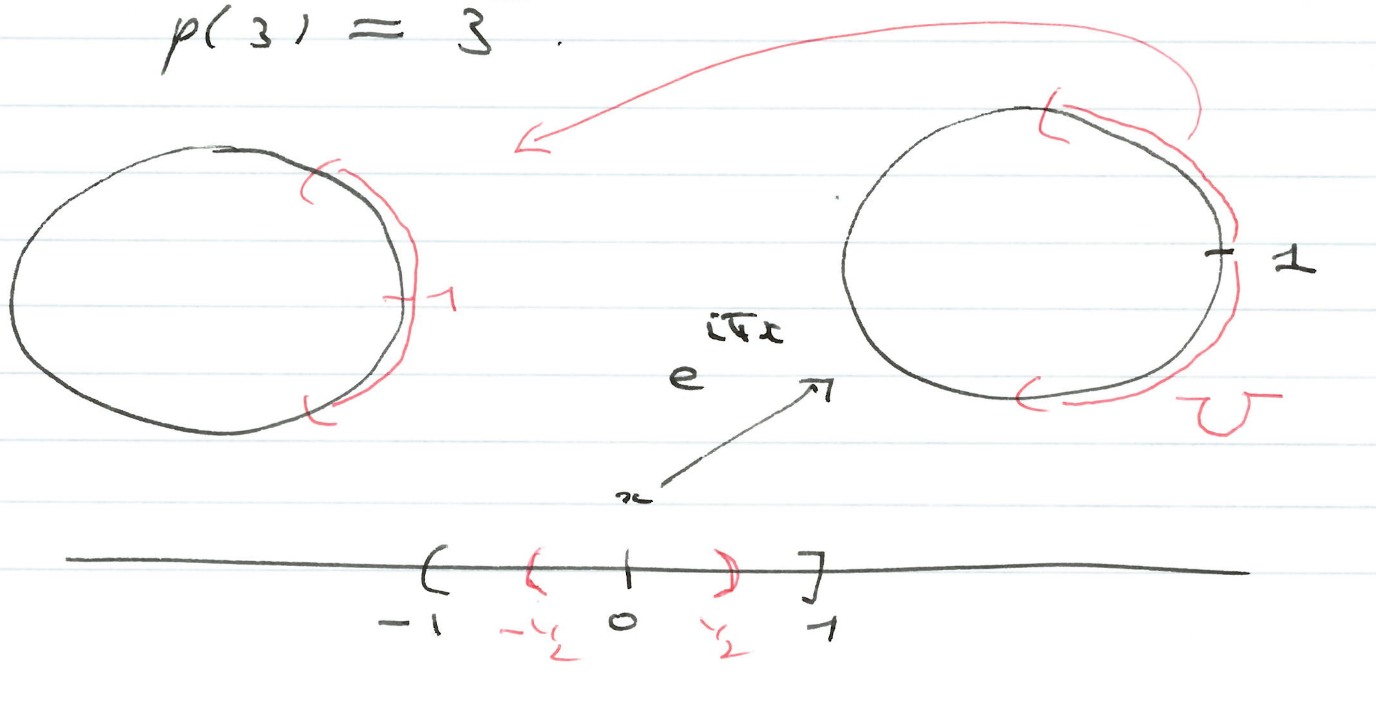
$p : \tilde{H} \longrightarrow H$ be a universal covering map, where we have fixed $e^* \in p^{-1}(e)$ and the corresponding topological group structure on $\tilde{H}$. Then $\mathrm{Ker}\, p = p^{-1}(e)$ being the fiber of e is a discrete subset of $\tilde{H}$ on one hand, and a normal subgroup in the connected group $\tilde{H}$ on the other hand, hence by Prop. 2-22 (v), $\mathrm{Ker}\, p \subset Z(\tilde{H})$; since $\pi_1(H, e) \simeq \mathrm{Ker}\, p$ this implies that $\pi_1(H, e)$ is abelian.

2.4. Local homomorphisms.

This section will be a cornerstone in establishing precisely the correspondence between Lie algebras and Lie groups. It concerns local homomorphisms; they occur for instance naturally as local sections near $e \in H$ of covering groups $p : G \to H$.

Ex 2.25: $G = H = \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$

$$p(z) = z^2 .$$



for $\xi = e^{i\pi x}$, $x \in (-1/2, 1/2)$

define $\varphi(\xi) := e^{\frac{i\pi x}{2}}$; then

$p \circ \varphi(\xi) = \xi \quad \forall \xi \in U$ and

for all $\xi_1, \xi_2 \in U$ with $\xi_1 \cdot \xi_2 \in U$

we have $\varphi(\xi_1 \cdot \xi_2) = \varphi(\xi_1)\, \varphi(\xi_2)$.

Def. 2.26 Let G, H be topological groups.

(1) A local homomorphism is a pair (φ, U) consisting of a neighborhood U of e and a continuous map $\varphi : U \to H$ such that whenever $\{x, y, xy\} \subset U$ then

$$\varphi(xy) = \varphi(x)\, \varphi(y).$$

(2) It is called a local isomorphism if $\varphi : U \to \varphi(U)$ is a homeomorphism.

<u>Example 2.27</u> If $p: G \to H$ is a covering homomorphism of top. groups then G and H are locally isomorphic: (exercise) explicit a local isomorphism from H to G.

<u>Thm 2.28</u> If $\varphi: U \to H$ is, $U \subset G$, a local homomorphism and G is pathwise connected and simply connected then φ extends (uniquely) to a continuous homomorphism

$$G \to H.$$

Here we will just give a sketch of proof. Details can be found in Tozzi 2.4 pp 16–19.

<u>Sketchy</u> <u>sketch</u> <u>of proof</u>:

The strategy is:

(1) For any continuous path $\alpha : [0,1] \to G$ with $\alpha(0) = e$, "extend φ along α" to define $\varphi(\alpha(1))$.

(2) Use that G is simply connected to show that if $\alpha_1, \alpha_2 : [0,1] \to G$ are continuous paths with $\alpha_1(0) = e = \alpha_2(0)$ and $\alpha_1(1) = \alpha_2(1)$ then $\varphi(\alpha_1(1)) = \varphi(\alpha_2(1))$.

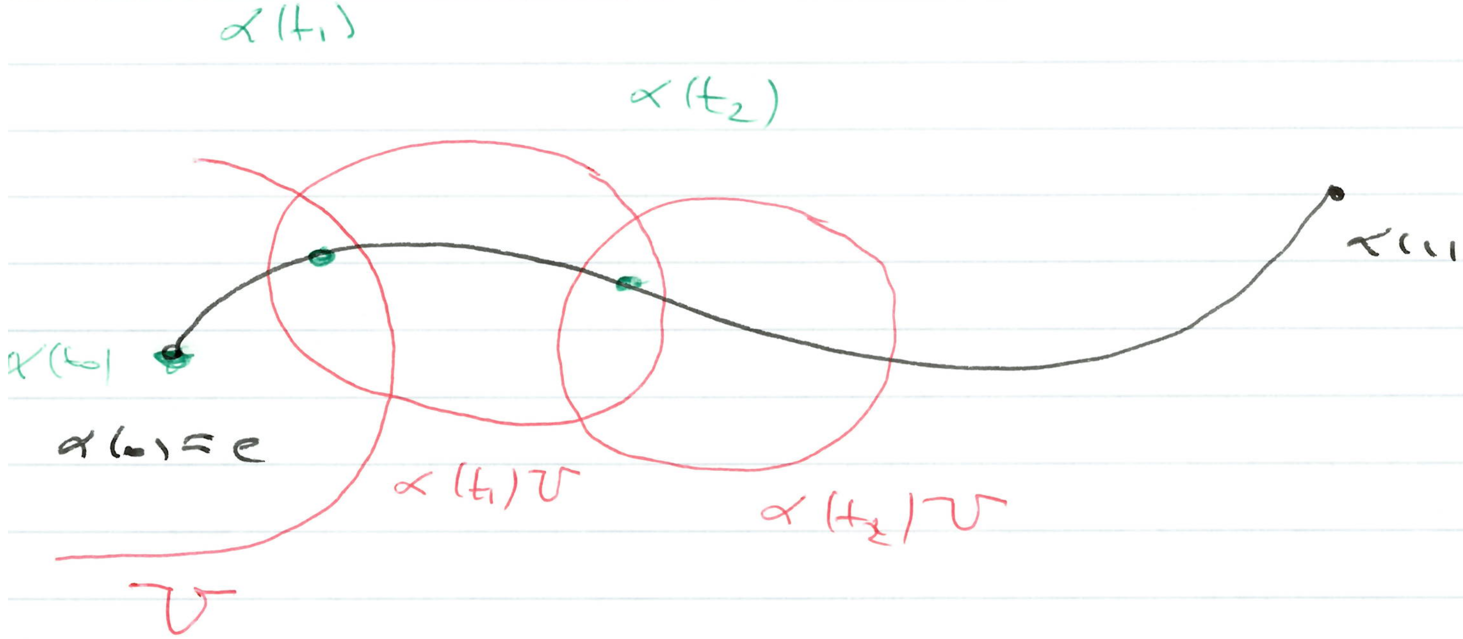
(3) Show that $\varphi : G \to H$ is continuous.

Concerning (1): let $\alpha : [0,1] \to G$ be continuous, $\alpha(0) = e$.

A partition $t_0 = 0 < t_1 < \dots < t_n = 1$

is <u>good</u> if $\forall s, t \in I_k := [t_{k-1}, t_k]$

$$\alpha(s)^{-1} \alpha(t) \in U.$$



Using that $\alpha([0,1])$ is compact one shows that good partitions always exist. Next:

- a refinement of a good partition is good.
- any two good partitions have a common refinement.

Given a good partition, observe that

$$\alpha(1) = \underbrace{\left(\alpha(t_0)^{-1}\alpha(t_1)\right)}_{\in\, \mathcal{V}}\underbrace{\left(\alpha(t_1)^{-1}\alpha(t_2)\right)}_{\in\, \mathcal{V}} \cdots \underbrace{\left(\alpha(t_{n-1})^{-1}\alpha(t_n)\right)}_{\in\, \mathcal{V}}$$

and define

$$\varphi(\alpha(1)) := \varphi\left(\alpha(t_0)^{-1}\alpha(t_1)\right) \cdots \varphi\left(\alpha(t_{n-1})^{-1}\alpha(t_n)\right).$$

If we refine the partition by adding one point, $\bar{t} \in (t_{k-1}, t_k)$

then:

$$\underbrace{\alpha(t_{k-1})^{-1}\alpha(t_k)}_{\in\, \mathcal{V}} = \underbrace{\alpha(t_{k-1})^{-1}\alpha(\bar{t})}_{\in\, \mathcal{V}}\,\underbrace{\alpha(\bar{t})^{-1}\alpha(t_k)}_{\in\, \mathcal{V}}$$

and hence since φ is a local homomorphism.

$$\varphi\left(\alpha(t_{k-1})^{-1}\alpha(t_k)\right) = \varphi\left(\alpha(t_{k-1})^{-1}\alpha(\bar{t})\right)\varphi\left(\alpha(\bar{t})^{-1}\alpha(t_k)\right).$$

This shows that if we use this refinement to define $\varphi(\alpha(1))$ we get the same result. This essentially shows that $\varphi(\alpha(1))$ is independent of the good partition.

(2) If now $\alpha_1, \alpha_2 \in : [0,1] \to G$ with $\alpha_1(0) = \alpha_2(0) = e$, $\alpha_1(1) = \alpha_2(1) = g$ are homotopic via a homotopy

$$H : [0,1] \times [0,1] \to G$$

with $H(0,t) = \alpha_1(t)$, $H(1,t) = \alpha_2(t)$, $t \in [0,1]$, let $W = W^{-1}$ open with $W^2 \subset U$ and let $\delta > 0$ with

$$H(s_1,t_1)^{-1} H(s_2,t_2) \in W$$

$\forall\ |s_1 - s_2| + |t_1 - t_2| < \delta$.

Let for $s \in [0,1]$, $\alpha_s : [0,1] \to G$

be $\alpha_s(t) = H(s,t)$ and choose

as partition $t_k = \frac{k}{n}$, $0 \leq k \leq n$,

with $\frac{1}{n} < \delta$.

Denoting $x_{s,k} := \alpha_s(t_k)$ we have

by construction

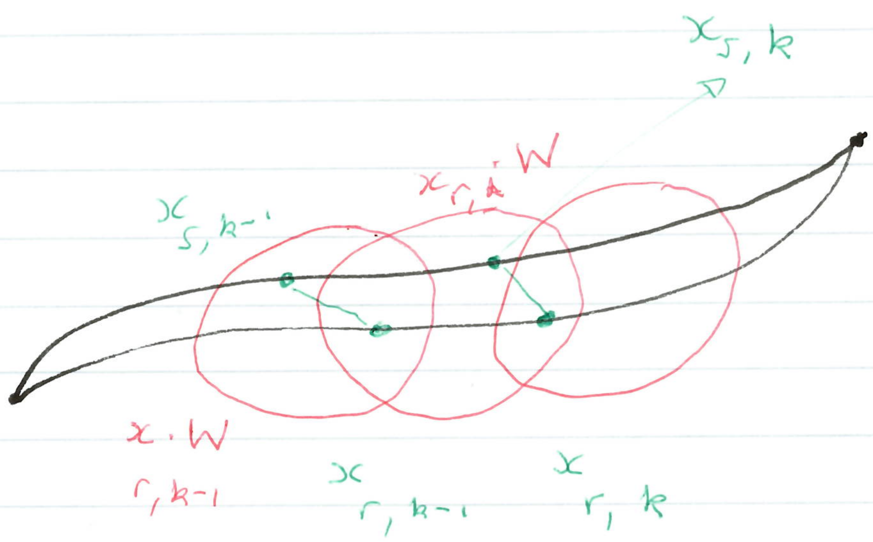
$$x_{s,k-1}^{-1} \cdot x_{s,k} \in W, \quad 1 \leq k \leq n$$
$$s \in [0,1].$$

Let then s, r be such that $|s-r| < \delta$

and consider the images of the partition

along the curves α_r and α_s; we

have then the following picture:



Since $|s-r|<\delta$ we have

$$y_k := x_{s,k}^{-1}\cdot x_{r,k} \in W .$$

Now we write

$$\underbrace{x_{s,k-1}^{-1}\cdot x_{s,k}}_{W} = \underbrace{y_{k-1}}_{W}\cdot \underbrace{\underbrace{(x_{r,k-1}^{-1}\cdot x_{r,k})}_{W}\,\underbrace{y_k^{-1}}_{W}}_{U}$$

Hence by the local hom. property:

$$\varphi(x_{s,k-1}^{-1}\cdot x_{s,k}) = \varphi(y_{k-1})\,\varphi\big((x_{r,k-1}^{-1}\cdot x_{r,k})\,y_k^{-1}\big)$$

$$= \varphi(y_{k-1})\,\varphi(x_{r,k-1}^{-1}\cdot x_{r,k})\,\varphi(y_k)^{-1}$$

Now we can compare the product defining $\varphi(\alpha_s(1))$ to the one defining $\varphi(\alpha_r(1))$:

$$\prod_{k=1}^{n} \varphi(x_{s,n-1}^{-1} \cdot x_{s,k}) =$$

$$= \varphi(x_{r,0}^{-1} x_{r,1}) \varphi(y_1)^{-1} \cdot \varphi(y_1) \varphi(x_{r,1}^{-1} \cdot x_{r,2}) \varphi(y_2) \cdot \dots$$

$$\dots \varphi(y_{n-1})^{-1} \varphi(y_{n-1}) \varphi(x_{r,n-1}^{-1} x_{r,n}) \varphi(y_n)^{-1}$$

Since $\varphi(y_0) = \varphi(y_n) = e$ and the $\varphi(y_k)^{-1} \cdot \varphi(y_k) = e$ $1 \leq k \leq n-1$ we get

$$\prod_{k=1}^{n} \varphi(x_{s,k-1}^{-1} x_{s,k}) = \prod_{k=1}^{n} \varphi(x_{r,k-1}^{-1} x_{r,k}).$$

This shows that $\varphi(g)$ is well defined independently of the path joining e to g.

The details of the remainder of the proof are left to the reader. $\square$

From Thm. 2.28 we conclude immediately

<u>Corollary 2.29</u>: Assume G is path conn., locally path conn., semi-locally simply connected, and let $p : \tilde{G} \longrightarrow G$ be a universal covering. Let

$$\begin{array}{c} G \\ \cup \\ \tilde{U} \xrightarrow[\varphi]{} H \end{array}$$

be a local homomorphism and $V \ni \tilde{e} \in \tilde{G}$ a neighborhood of $\tilde{e}$ with $p(V) \subset \tilde{U}$. Then $p \circ \varphi : V \longrightarrow H$ extends to a unique continuous homomorphism

$$\tilde{G} \longrightarrow H.$$