2.5. Haar measure and homogeneous spaces.

This section concentrates on locally compact Hausdorff groups, (henceforth called l.c.h. groups); the fundamental fact is the existence and uniqueness of Haar measure and Weil's criterion for the existence of invariant measures on homogeneous spaces of l.c.h. groups.

2.5.1. Haar measure.

Let's fix some terminology: a left action of a top. group G on a top. space X is continuous if the action map $G \times X \to X$ is continuous.

Given any map $F: X \to Y$ we let $(\lambda(g)F)(x) := F(g^{-1}x)$; when G and X are l.c.h., and $C_{00}(X)$ is the space of compactly supported continuous functions, then $\forall g \in G$

$$\lambda(g): C_{00}(X) \to C_{00}(X)$$

is an endomorphism and $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$.

Recall (see Rudin, "Real and Complex Analysis" Thm 2.14)

<u>Thm 2.30.</u> (Riesz Repr.) Let X be l.c.h. $\Lambda: C_{00}(X) \to \mathbb{C}$ a positive linear functional. Then there is a σ-Algebra $\mathfrak{M}$ containing all Borel subsets of X and a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense:

(a) $\Lambda(f) = \int_X f(x)\,d\mu(x) \quad \forall f \in C_{00}(X)$

with the additional properties

(b) $\mu(K) < +\infty \quad \forall K \subset X$ compact

(c) $\forall E \in \mathfrak{M}: \mu(E) = \inf\{\mu(V): E \subset V, V \text{ open}\}$

(d) $\mu(E) = \sup\{\mu(K): K \subset E, K \text{ compact}\}$

holds $\forall E$ open and $\forall E \in \mathfrak{M}$ with $\mu(E) < +\infty$.

~~(e) If E ∈ M and A ⊂ E~~

(e) If $E \in \mathfrak{M}$, $A \subset E$ and $\mu(E) = 0$ then $A \in \mathfrak{M}$.

Positive means that if $f \in C_{00}(X)$ with $f(X) \subset [0, \infty[$ then $\Lambda(f) \geq 0$.

If now $G \times X \to X$ is a continuous left action and $\Lambda \in C_{00}(X)^*$ is a linear functional, we define

$(\lambda^*(g)\Lambda)(f) := \Lambda(\lambda(g)^{-1}f)$ and

obtain an endomorphism $\lambda^*(g) : C_{00}(X) \to$

$\to C_{00}(X)$ with $\lambda^*(g_1 g_2) = \lambda^*(g_1)\lambda^*(g_2)$.

If now Λ is a positive linear functional

and μ the corresponding regular Borel

measure then $\lambda^*(g)\Lambda$ is represented

by the measure $g_*\mu$ where:

$$(g_*\mu)(A) := \mu(g^{-1}A) .$$

(Exercise).

<u>Definition 2.31</u> A left (invariant)

Haar functional on a l.c.h. group G

is a positive, non-zero, functional

$$\Lambda : C_{00}(G) \longrightarrow \mathbb{C}$$

such that $\lambda^*(g)\Lambda = \Lambda \quad \forall g \in G$.

The associated regular Borel measure μ is called a left (invariant) Haar measure.

One defines analogously a right (invariant) Haar functional and a right (invariant) Haar measure.

The fundamental theorem is then:

Thm 2.32 (Haar 1933) Let G be a l.c.h. group. Then there exists a left Haar functional and it is unique up to multiplication by an element of $\mathbb{R}_{>0}$.

Thus:

Corollary 2.33: There exists, up to a unique scalar multiple in $\mathbb{R}_{>0}$, a positive regular Borel measure μ on G such that

For every measurable $E \subset G$ and $g \in G$:

$$\mu(gE) = \mu(E).$$

Exercise: let for $f \in C_{00}(G)$, $g \in G$ $(\rho(g)f)(x) = f(xg)$. Then $\rho(g) \in \operatorname{End} C_{00}(G)$ and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. Let $\Lambda \in C_{00}(G)^*$ and define $(\rho^*(g)\Lambda)(f) = \Lambda(\rho(g)^{-1}f)$. Then Λ is a right (invariant) Haar functional if Λ is positive and $\rho^*(g)\Lambda = \Lambda$ $\forall g \in G$. The corresponding measure μ is called a right (inv.) Haar measure. Let $\check{f}(x) := f(x^{-1})$. Show that if Λ is a left Haar functional, then

$$\Lambda'(f) := \Lambda(\check{f})$$ is a right Haar functional. Thus Thm 2.32 implies the existence of a right Haar functional and a right Haar measure.

Examples 2.34.

(1) The Lebesgue measure on $(\mathbb{R}^n, +)$ is a left and right Haar measure.

(2) Let λ be Lebesgue measure on $\mathbb{R}$. Then for $f \in C_{00}(\mathbb{R}_{>0})$,

$$I(f) = \int_0^\infty f(x) \frac{d\lambda(x)}{x}$$

is a left and right Haar measure on $(\mathbb{R}^\times_{>0}, \cdot)$.

(3) G a discrete group, then counting measure is a left and right Haar measure.

(4) Let $P = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : \begin{matrix} x \in \mathbb{R}^\times_{>0} \\ y \in \mathbb{R} \end{matrix} \right\}$

And for $f \in C_{00}(P)$:

$$I(f) = \int_0^\infty \frac{d\lambda(x)}{x^2} \int_{-\infty}^\infty d\lambda(y)\, f(x,y)$$

Then if $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$,

$$(\lambda(g)f)(x,y) = (a^{-1}x, a^{-1}y - bx^{-1})$$

Then $\displaystyle\int d\lambda(y)\, f(a^{-1}x, a^{-1}y - bx^{-1})$

$$= a \int d\lambda(y)\, f(a^{-1}x, y - bx^{-1})$$

$$= a \int d\lambda(y)\, f(a^{-1}x, y)$$

So $\displaystyle\int_0^\infty \frac{d\lambda(x)}{x^2} \int_{-\infty}^\infty d\lambda(y)\, (\lambda(g)f)(x,y)$

$$= a \int_0^\infty \frac{d\lambda(x)}{x^2} \int d\lambda(y)\, f(a^{-1}x, y)$$

$$= \int_0^\infty \frac{d\lambda(x)}{x^2} \int d\lambda(y)\, f(x,y) = I(f).$$

While: with $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$

$$I(\rho(g) f) = \int_0^\infty \frac{d\lambda(x)}{x^2} \int_{-\infty}^\infty d\lambda(y) f(ax, bx + a^{-1}y)$$

Then $\displaystyle\int_{-\infty}^\infty d\lambda(y) f(ax, bx + a^{-1}y)$

$$= a \int_{-\infty}^\infty d\lambda(y) f(ax, y + bx)$$

$$= a \int_{-\infty}^\infty d\lambda(y) f(ax, y)$$

Hence

$$I(\rho(g) f) = a \int_0^\infty \frac{d\lambda(x)}{x^2} \int_{-\infty}^\infty d\lambda(y) f(ax, y)$$

$$= a^2 \int_0^\infty \frac{d\lambda(x)}{x^2} \int_{-\infty}^\infty d\lambda(y) f(x, y)$$

$$= a^2 I(f).$$

Thus $\dfrac{d\lambda(x)}{x^2} d\lambda(y)$ is a left Haar measure on $\mathcal{L}$ but not right Haar measure.

(5) Haar measure on $GL(n, \mathbb{R})$:

let $dm(x) = \prod_{i,j} d\lambda(x_{i,j})$, on $M_{n,n}(\mathbb{R})$ restricted to the open subset $GL(n, \mathbb{R})$. We claim that for $f \in C_{00}(GL(n, \mathbb{R}))$,

$$I(f) = \int_{GL(n,\mathbb{R})} f(x) \frac{dm(x)}{|\det x|^n}$$

is a left and right Haar measure.

Now recall the formula for change of variables in $\mathbb{R}^m$: if $\Omega \subset \mathbb{R}^{\cancel{m}\,N}$ is open, $\Psi : \Omega \to \Omega$ a diffeomorphism and say m the Lebesgue measure on $\mathbb{R}^N$ then $\forall F \in C_{00}(\Omega)$:

$$(*) \quad \int_\Omega F(\psi(\omega))\,|\det D_\omega \psi|\, dm(\omega) = \int_\Omega F(x)\, dm(x)$$

Now let $\Omega = GL(n, \mathbb{R})$

$$\psi(\omega) = g\cdot\omega \quad , \quad \omega \in GL(n,\mathbb{R})$$

$$F(x) = \frac{f(x)}{|\det x|^n} \quad , \quad f \in C_0(GL(n,\mathbb{R})).$$

Then $\quad (D_\omega \psi)(X) = g\cdot X$

since $\quad \omega \mapsto g\cdot\omega$ is a linear map

on $M_{n,n}(\mathbb{R})$ and:

$$\det D_\omega \psi = (\det g)^n .$$

Thus we obtain from $(*)$:

$$\int_{GL(n,\mathbb{R})} \frac{f(g\omega)}{|\det g\omega|^n}\, |\det g|^n\, dm(\omega)$$

$$= \int_{GL(n,\mathbb{R})} \frac{f(x)}{|\det x|^n}\, dm(x).$$

That is:

$$\int_{GL(n,\mathbb{R})} \frac{f(gw)}{|\det w|^n}\, dm(w) = \int_{GL(n,\mathbb{R})} \frac{f(x)}{|\det x|^n}\, dm(x).$$

A similar calculation with

$$\psi(w) = w g$$

shows that $\dfrac{dm(x)}{|\det x|^n}$ is also

a right Haar measure on $GL(n,\mathbb{R})$.