(6) Let $N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

be the Heisenberg group. Then we claim that, using

$$\mathbb{R}^3 \to N$$
$$(x, y, z) \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

as parametrisation of N, for $f \in C_{00}(N)$

$$I(f) = \int_{\mathbb{R}^3} f(x, y, z)\, d\lambda(x)\, d\lambda(y)\, d\lambda(z)$$

is a left and right Haar measure.

Indeed: let $g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

then $g \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & x+a & z+c+ay \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$I(\lambda(g^{-1})f) = \int_{\mathbb{R}^3} f(x+a,\, y+b,\, z+c+ay)\, d\lambda(x)\, d\lambda(y)\, d\lambda(z)$$

Since

$$\int_{\mathbb{R}} d\lambda(z)\, f(x+a,\, y+b,\, z+c+ay)$$

$$= \int_{\mathbb{R}} d\lambda(z)\, f(x+a,\, y+b,\, z)$$

we get

$$I(\lambda(g^{-1})f) = \int_{\mathbb{R}^3} f(x+a,\, y+b,\, z)\, d\lambda(x)\, \lambda(y)\, d\lambda(z)$$

$$= \int_{\mathbb{R}^3} f(x,\, y,\, z)\, d\lambda(x)\, d\lambda(y)\, d\lambda(z)$$

$$= I(f)$$

A similar computation shows right invariance

Now we are going to exploit the uniqueness statement in Haar's theorem. Let

$$\mathrm{Aut}(G) = \{ \alpha : G \to G \text{, is a group automorphism such that } \alpha \text{ and } \alpha^{-1} \text{ are continuous} \}.$$

which is clearly a group under composition; we already saw special kinds of such automorphisms, namely

$$\mathrm{int}(g) : G \longrightarrow G$$
$$x \longmapsto g x g^{-1}$$

called interior automorphisms.

Define for $\alpha \in \mathrm{Aut}\, G$ and $f \in C_{oo}(G)$,

$$(\alpha . f)(x) := f(\alpha^{-1} x)$$

then this defines a left action of $\mathrm{Aut}\, G$ on $C_{oo}(G)$.

Let I be a left Haar functional and define $I_\alpha(f) := I(\alpha_* f)$, $f \in C_{00}(G)$

We claim that I_α is a left Haar functional: we compute for $g \in G$

$$(\alpha_* \lambda(g))(f)(x) = f(g^{-1}\alpha^{-1}(x))$$

$$= f(\alpha^{-1}[\alpha(g)^{-1}(x)]) = (\alpha_* f)(\alpha(g)^{-1} x)$$

$$= \lambda(\alpha(g))(\alpha_* f)(x)$$

Hence $I_\alpha(\lambda(g) f) = I(\alpha_* \lambda(g) f)$

$$= I(\lambda(\alpha(g))(\alpha_* f))$$

$$= I(\alpha_* f) = I_\alpha(f).$$

Thus there exists a positive scalar denote $\mathrm{mod}_G(\alpha) \in \mathbb{R}^\times_{>0}$ with

$$I_\alpha(f) = \mathrm{mod}_G(\alpha)\, I(f) \quad \forall f \in C_{00}(G)$$

<u>Lemma 2.35</u> $\mathrm{mod}_G : \mathrm{Aut}\, G \longrightarrow \mathbb{R}^*_{>0}$ is a homomorphism.

<u>Proof</u>: The point here is that $\mathrm{mod}_G(\alpha)$ is <u>independent</u> on the choice of left Haar functional I: indeed any other Haar functional I' satisfies $I' = c \cdot I$ for some $c > 0$ so that $I'_\alpha = c \cdot I_\alpha = c \cdot \mathrm{mod}_G(\alpha) I = \mathrm{mod}_G(\alpha) I'$.

With this we have $\forall \alpha, \beta \in \mathrm{Aut}\, G$:

$$I_{\alpha\cdot\beta}(f) = I(\beta_*^{-1}\alpha_*^{-1}(f))$$

$$= I_\beta(\alpha_*^{-1}(f))$$

$$= \mathrm{mod}_G(\beta) I(\alpha_*^{-1}(f))$$

$$= \mathrm{mod}_G(\beta) \mathrm{mod}_G(\alpha) I(f)$$

And $I_{\alpha\cdot\beta}(f) = \mathrm{mod}_G(\alpha\cdot\beta) I(f)$. $\boxed{\text{S}}$

Applying this in particular to $\mathrm{int}(g)$, let's define the <u>modular</u> function Δ_G of G by $\Delta_G(g) = \mathrm{mod}_G(\mathrm{int}(g))$.

If μ is a left Haar measure we obtain from left invariance of μ:

$$\Delta_G(g) \int_G f(x)\, d\mu(x) = \int_G f(g^{-1}xg)\, d\mu(x)$$

$$= \int_G f(xg)\, d\mu(x)$$

and thus Δ_G measures the extent to which μ is not right invariant. We have

<u>Prop. 2.36</u> (i) $\Delta_G : G \to \mathbb{R}^\times_{>0}$ is a continuous homomorphism.

(2) $\forall f \in C_{00}(G)$:

$$\int_G f(x^{-1})\, \Delta_G(x)\, d\mu(x) = \int_G f(x)\, d\mu(x).$$

We will need a few preliminary facts of independent interest:

<u>Def. 2.37</u> A map $F : G \to Y$, where (Y, d) is a metric space, is left uniformly continuous (resp. right uniformly continuous) if $\forall \varepsilon > 0$ there is a neighborhood $U \ni e$ such that $d(f(x), f(y)) < \varepsilon$ $\forall (x,y)$ with $x^{-1} y \in U$ (resp. $x y^{-1} \in U$).

The terminology comes from the fact that if F is left uniformly continuous and

$d(F(x), F(y)) < \varepsilon$ whenever $x^{-1}y \in U$

then $d(F(gx), F(gy)) < \varepsilon$ $\forall g \in G$

since $(gx)^{-1}gy = x^{-1}g^{-1}gy = x^{-1}y \in U$.

Lemma 2.38 (1) If $f \in C_{00}(G)$, then $f : G \longrightarrow \mathbb{C}$ is left and right uniformly continuous.

(2) If we endow $C_{00}(G)$ with the distance coming from $\|f\|_\infty := \sup_{x \in G} |f(x)|$ then $\forall f \in C_{00}(G)$, the maps

$$\begin{aligned} G &\longrightarrow C_{00}(G) \\ g &\longmapsto \lambda(g) f \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow C_{00}(G) \\ g &\longmapsto \rho(g) f \end{aligned}$$

are continuous.

Observe that (2) is litterally equivalent to (1).

<u>Proof</u> (of (1) in the case of left uniform continuity).

Let $V = V^{-1} \ni e$ be open with $\overline{V}$ compact and let $K := (\operatorname{supp} f) \cdot \overline{V}$ which is compact (exercise).

If x, y are such that $x^{-1}y \in V$ and $x \notin K$, then from $y \in xV$ we get with $xV \cap \operatorname{supp} f = \emptyset$ that $f(x) = f(y) = 0$. Similarly if $y \notin K$ using that $V = V^{-1}$.

Now $\forall x \in K$, $\exists W_x \ni e$ neighborhood with $W_x \subset V$ and

$$|f(x) - f(y)| < \varepsilon/2 \quad \forall y \in K \cap (x \cdot W_x).$$

Now let $V_x \ni e$ open with $V_x^{-1} = V_x$ and $V_x^2 \subset W_x$. Since $K \subset \bigcup_{x \in K} x V_x$

there exist $x_1, \dots, x_n$ with

$$K \subset \bigcup_{i=1}^{n} x_i V_{x_i} .$$

Let $V := \bigcap_{i=1}^{n} V_{x_i}$; if $x^{-1} \cdot y \in V$, $x \in K$ and $y \in K$ let $1 \le i \le n$ be such that $x \in x_i V_{x_i}$. Then:

$y \in x \cdot V \subset x_i V_{x_i}^2 \subset x_i W_{x_i}$ and hence

$$|f(x) - f(y)| \le |f(x) - f(x_i)| + |f(x_i) - f(y)|$$

$$< \varepsilon .$$

If $x^{-1} \cdot y \in V \subset U$ and either $x \notin K$ or $y \notin K$ then we have seen above that $f(x) = 0 = f(y)$. $\boxed{?}$

<u>Lemma 2.39</u>: Let μ be a left Haar measure on G.

(1) If $\Omega \subset G$ is open, non empty then $\mu(\Omega) > 0$; equivalently the support of $\mu = G$.

(2) Assume $F_1, F_2 : G \longrightarrow Y$ are continuous maps into a top Hausdorff space such that $F_1(g) = F_2(g)$ for μ-almost every $g \in G$.

Then $F_1(g) = F_2(g)$ $\forall g \in G$.

(3) ~~Let $\Omega \subset G$ open and with $\operatorname{supp} f \subset \Omega$~~ Let $f \in C_{00}(G)$ with $0 \leq f \leq 1$ and $f \neq 0$, then $\int_G f(x)\, d\mu(x) > 0$.

Proof: (1) By definition a left Haar-measure μ is not identically zero. This means, there exists $f \in C_{00}(G)$ which we may assume $0 \leq f \leq 1$ and $\int_G f(x)\,d\mu(x) > 0$.

Now since supp f is compact there exist $g_1, \dots, g_n$ with

$$\operatorname{supp} f \subset \bigcup_{i=1}^{n} g_i \Omega$$

which implies $f(x) \leq \|f\|_\infty \sum_{i=1}^{n} \chi_{g_i\Omega}(x)$

and hence

$$0 < \int_G f(x)\,d\mu(x) \leq \|f\|_\infty \sum_{i=1}^{n} \int_G \chi_{g_i\Omega}(x)\,d\mu(x)$$

$$= n \cdot \mu(\Omega) \cdot \|f\|_\infty$$

which shows (1).