Assertions (2) and (3) are easy consequences of (1). $\boxed{\xi}$

Proof of Prop. 2.36

(1) Since $\Delta_G : G \longrightarrow \mathbb{R}^\times_{>0}$ is a homomorphism it suffices to show continuity at e. Pick $f \in C_{oo}(G)$ with $f \geq 0$ and $\int_G f(\xi) d\mu(\xi) = 1$.

Then

$$|(\Delta_G(g) - 1) \cdot 1| = |\int_G (f(\xi g) - f(\xi)) d\mu(\xi)|$$

We may assume $g \in U = U^{-1}$ where $U \ni e$ is open with $\overline{U}$ compact.

Let $K := (\operatorname{supp} f) \cdot \overline{U}$ and observe that if $\xi \notin K$ then $f(\xi g) = f(\xi) = 0$.

Hence

$$\left| \int_G f(\xi g) - f(\xi) \, d\mu(\xi) \right|$$

$$\leq \mu(K) \cdot \| \rho(g) f - f \|_\infty .$$

Then the continuity at $g = e$ of Δ_G follows from the continuity at $g = e$ of

$$G \to (C_{00}(G), \| \ \|_\infty)$$
$$g \mapsto \rho(g) f$$

which follows from lemma 2.38 (2).

(2) Since Δ_G is continuous, we have that $\forall f \in C_{00}(G)$, $f \cdot \Delta_G \in C_{00}(G)$.

Define $I(f) := \int_G f(x^{-1}) \Delta_G(x) \, d\mu(x)$.

Then $I(\lambda(g) f) = \int_G f(g^{-1} x^{-1}) \Delta_G(x) \, d\mu(x)$

$$= \int_G f((xg)^{-1})\, \Delta_G(xg)\, d\mu(x)\, \Delta_G(g)^{-1}$$

$$= \int_G f(x^{-1})\, \Delta_G(x)\, d\mu(x)\, \Delta_G(g)\, \Delta_G(g)^{-1}$$

$$= I(f).$$

Thus I is a left Haar functional and hence there is $c > 0$ with

$$\int_G f(x^{-1})\, \Delta_G(x)\, d\mu(x) = c \cdot \int_G f(y)\, d\mu(y).$$

Write $\int_G f(y)\, d\mu(y) = \int_G \left(f(y)\, \Delta_G(y^{-1})\right) \Delta_G(y)\, d\mu(y)$

and define $F(y) = f(y^{-1})\, \Delta_G(y)$.

Then $\int_G f(y)\, d\mu(y) = \int_G F(y^{-1})\, \Delta_G(y)\, d\mu(y)$

$$= c \cdot \int_G F(y)\, d\mu(y) = c \cdot \int_G f(y^{-1})\, \Delta_G(y)\, d\mu(y)$$

$$= c^2 \int_G f(z)\, d\mu(z)$$

implying $\int_G f(y)\, d\mu(y) = c^2 \int_G f(z)\, d\mu(z)$

and hence, since $c > 0$, $c = 1$.

$\boxed{\xi}$

<u>Def. 2.40</u> A l.c.h. group G is unimodular if $\Delta_G(g) = 1$ $\forall g \in G$; equivalently every left Haar measure is a right Haar measure.

<u>Examples 2.41.</u> In Examples 2.34 we have that $(\mathbb{R}^n, +)$, $(\mathbb{R}^\times_{>0}, \cdot)$, discrete groups, $GL(n, \mathbb{R})$, the Heisenberg group, are all unimodular. While $P = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : \begin{array}{l} x > 0 \\ y \in \mathbb{R} \end{array} \right\}$

is not and $\Delta_\rho \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} = x^2$.

(see Example 2.34 (41)).

We end this section by characterizing l.c.a. groups of finite Haar measure, underlying that properties of the Haar measure are linked to topological properties of the group.

~~Example 2.42~~

Prop. 2.42. A l.c.h group G is compact iff it has finite Haar measure.

Proof: Let μ be a left Haar measure on G.

$(\Rightarrow)$ If G is compact, then by regularity of μ we have $\mu(G) < +\infty$.

($\Leftarrow$) Assume $\mu(G) < +\infty$. Since μ is regular there is $K \subset G$ compact with $\mu(K) > \frac{1}{2}\mu(G)$. Let $g \in G$: then $\mu(gK) = \mu(K)$ and hence $\mu(gK) + \mu(K) > \mu(G)$ which implies that $gK \cap K \neq \emptyset$ and hence $g \in KK^{-1}$. Thus $G = KK^{-1}$ and is hence compact.

$\boxed{\Xi}$

2.5.2. Homogeneous Spaces.

Homogeneous spaces of locally compact groups in general and Lie groups in particular give extremely rich playgrounds for geometry, topology and dynamics.

Here we just discuss the basics and the existence of invariant measures. For the basic definitions we place ourselves in the context of general topological groups. Let G be a topological group, $H < G$ a subgroup and G/H the set ~~of~~

$$\{ x \cdot H \; : \; x \in G \}$$

of right H-cosets. Let $p : G \to G/H$ be the canonical projection. We endow G/H with the quotient topology, that is $U \subset G/H$ is open iff $p^{-1}(U) \subset G$ is open.

Prop. 2.43 Let $H < G$ be a subgroup of a top. group G.

(1) The projection map $p : G \longrightarrow G/H$ is open.

(2) The action $G \times G/H \longrightarrow G/H$ is continuous.

(3) The quotient G/H is Hausdorff iff H is closed.

(4) If G is locally compact so is G/H

(5) If G is locally compact and $H < G$ is closed then $\forall$ $C \subset G/H$ compact there is $K \subset G$ compact such that $p(K) = C$.

Proof: We leave (1) and (2) as exercises.

(3) If G/H is Hausdorff, then points in G/H are closed, in particular H is closed.

Assume H closed; if $xH \neq yH$ then $xHy^{-1} \not\ni e$ and since xHy^{-1} is closed, there is $V \ni e$ open with $V^{-1} V \cap xHy^{-1} = \phi$.

Which is equivalent to $Vy \cap VxH = \phi$ and hence $VyH \cap VxH = \phi$. Thus $p(Vy)$ and $p(Vx)$ are open neighborhoods resp. of yH and xH with $p(Vy) \cap p(Vx) = \phi$.

(4) We have to show that every point in G/H has a compact neighborhood. Now let $p(x) \in G/H$, and since G is locally compact let $x \in U \subset C \subset G$ with U open and C compact; then $p(x) \in p(U) \subset p(C)$; $p(U)$ is open (by (1)) and $p(C)$ is compact.

(5) Let $e \in V \subset L$ where V is open and L is compact. Then $\bigcup_{x \in G} p(V \cdot x) \supset C$ is an open covering of C hence there are $x_1, \dots, x_n$ in G with $\bigcup_{i=1}^{n} p(V \cdot x_i) \supset C$. This implies

$$\bigcup_{i=1}^{n} p(L \cdot x_i) \supset C.$$

Since H is closed, G/H is Hausdorff (see (3)) and hence C being compact is closed. But then $K := p^{-1}(C) \cap \left(\bigcup_{i=1}^{n} p(L \cdot x_i)\right)$ being a closed subset of the compact set $\bigcup_{i=1}^{n} p(L \cdot x_i)$ is compact and

$$p(K) = C.$$

$\boxed{5}$

Examples
~~Exercise~~ 2.44.

(1) In $\mathbb{R}^n$ with the standard scalar product

let $1 \leq k \leq n$ and

$$GO_k := \{(v_1, \ldots, v_k) \in (\mathbb{R}^n)^k : v_1, \ldots, v_k \text{ is an orthonormal set}\}.$$

Then for every $g \in O(n, \mathbb{R})$ and

$(v_1, \ldots, v_k) \in GO_k$, $(g(v_1), \ldots, g(v_k)) \in GO_k$

and $O(n, \mathbb{R})$ acts transitively on GO_k.

In fact if $1 \leq k \leq n-1$ then $SO(n, \mathbb{R})$

acts transitively. The stabilizer of

$(e_1, \ldots, e_k)$ in $O(n, \mathbb{R})$ is:

$$L_m = \left\{ \begin{pmatrix} Id_k & 0 \\ 0 & M \end{pmatrix} : M \in O(n-k, \mathbb{R}) \right\}$$

$$\simeq O(n-k, \mathbb{R})$$

and hence
$$O(n,\mathbb{R})/L_k \longrightarrow GO_k$$
$$g\,L_k \longmapsto (g(e_1), \ldots, g(e_k))$$

is a bijection which allows to put on GO_k a topology of compact Hausdorff space on which $O(n,\mathbb{R})$ acts continuously.

(2) Let d be an ordered r-tuple:
$$1 \leq d_1 < d_2 < \cdots < d_r < n$$

and
$$Gr_d := \{ (V_1, V_2, \ldots, V_r) : V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{R}^n$$
with $\dim V_i = d_i \}$.

Then $GL(n,\mathbb{R})$ acts transitively on Gr_d and if $E_i = \mathbb{R}e_1 + \cdots + \mathbb{R}e_{d_i}$ then if $E = (E_1, E_2, \ldots, E_r)$

$$\operatorname{Stab}_{GL(n,\mathbb{R})}(E) = \left\{ \begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & & * \\ 0 & 0 & \ddots & \\ 0 & \cdots & 0 & A_r \end{pmatrix} \right\}$$

where $A_1 \in GL(d_1, \mathbb{R})$, $A_2 \in GL(d_2 - d_1, \mathbb{R})$ $\cdots$ $A_r \in GL(d_r - d_{r-1}, \mathbb{R})$.

Let us denote by $P_{\underline{d}} = \operatorname{Stab}_{GL(n,\mathbb{R})}(E)$.

Clearly $P_{\underline{d}} \supset P_{(1,2,\ldots,n-1)}$ the latter being the subgroup of upper triangular matrices.

<u>Exercise</u>: show that $GL(n,\mathbb{R}) / P_{(1,2,\ldots,n-1)}$ is compact and deduce that $GL(n,\mathbb{R}) / P_{\underline{d}}$ is compact. Hint:

Observe that

$$Gr_{(1,2,\dots,n-1)} = \left\{ (V_1, \dots, V_{n-1}) : V_1 \subset V_2 \subset \dots \subset V_{n-1} \atop \dim V_j = j \right\}$$

~~and that~~ also called the grassmannian of complete flags. Observe that for any complete flage $V_1 \subset \dots \subset V_{n-1}$ we can find an adapted orthonormal basis $(e_1, \dots, e_n)$, that is

$$V_i = \mathbb{R}e_1 + \dots + \mathbb{R}e_i .$$

Then use example (c) to show that $O(n, \mathbb{R})$ acts transitively on $Gr_{(1,2,\dots,n-1)}$.

(3) A lattice (ad hoc def.) in $\mathbb{R}^n$ is a discrete subgroup $\Lambda < \mathbb{R}^n$ such that $\mathbb{R}^n/\Lambda$ is compact. A basic example is

$$\Lambda_0 = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n.$$

Let $\mathcal{R} = \{ \Lambda < \mathbb{R}^n : \Lambda \text{ is a lattice} \}$.

Then if $g \in GL(n,\mathbb{R})$ and $\Lambda \in \mathcal{R}$, we have $g(\Lambda) \in \mathcal{R}$ (Exercise).

Moreover if $\Lambda \in \mathcal{R}$ then there is a basis $f_1, \dots, f_n$ of $\mathbb{R}^n$ such that

$$\Lambda = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_n \quad \text{(exercise)}.$$

Thus $GL(n,\mathbb{R})$ acts transitively on $\mathcal{R}$.

The stabilizer of Λ_0 is

$$GL(n,\mathbb{Z}) := \{ A \in M_{n,n}(\mathbb{Z}) : \det A = \pm 1 \}.$$

As a result $GL(n,\mathbb{R})/_{GL(n,\mathbb{Z})} \longrightarrow \mathcal{R}$

$$[g] \longmapsto g\Lambda_0$$

is a bijection. This can be used to put a topology on $\mathcal{R}$. Observe incidentally that $GL(n,\mathbb{Z})$ is a discrete subgroup of $GL(n,\mathbb{R})$.

(4) $SL(2,\mathbb{R})$ acts on the Poincaré upper half plane

$$\mathbb{H}^2_{\mathbb{R}} = \{ z \in \mathbb{C} : z = x+iy,\ y>0 \}$$

by $g_* z = \dfrac{az+b}{cz+d}$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : \begin{matrix} a>0 \\ b\in\mathbb{R} \end{matrix} \right\}$

acts simply transitively, as:

$$\begin{pmatrix} y^{1/2} & x y^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \cdot i = x + iy$$

and the stabilizer of i is

$$\mathrm{Stab}_{SL(2,\mathbb{R})}(i) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}$$

$$= SO(2, \mathbb{R}).$$

Now we turn to the following question:

Let G be l.c.h., $H < G$ a closed subgroup so that G/H is a l.c.h. space with continuous G-action.

Q: When does there exist a G-invariant positive functional on $C_{00}(G/H)$, equiv. when does there exist a G-invariant (positive) regular Borel measure on G/H?

Exercise

(1) $SL(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ and the orbit map

$$SL(2,\mathbb{R})/N \longrightarrow \mathbb{R}^2 \setminus \{0\}$$
$$gN \longmapsto g(e_1)$$

where $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$ is a $SL(2,\mathbb{R})$-equivariant homeo. Using this show that there is an $SL(2,\mathbb{R})$-invariant regular Borel measure on $SL(2,\mathbb{R})/N$.

(2) $SL(2,\mathbb{R})$ acts transitively on $\mathbb{P}^1(\mathbb{R})$ the projective line and

$$B = \mathrm{Stab}(\mathbb{R}e_1) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : \begin{array}{l} a \in \mathbb{R}^\times \\ b \in \mathbb{R} \end{array} \right\}.$$

Show that on $SL(2,\mathbb{R})/B$ there is no $SL(2,\mathbb{R})$ - invariant regular Borel measure.

The following gives a complete answer:

<u>Thm 2.45</u> Let G be l.c.h with left Haar measure μ_G, $H < G$ a closed subgroup with left Haar measure μ_H. Then there is a G-invariant (positive) regular Borel measure on G/H iff

$$\Delta_G \big|_H = \Delta_H .$$

In this case, said regular Borel measure is unique up to positive scalar multiple and there is a unique choice $\mu_{G/H}$

such that Weil's formula holds:

$$\int_G f(g)\,d\mu(g) = \int_{G/H} \int_H f(gh)\,d\mu_H(h)\,d\mu_{G/H}(gH)$$

$\forall f \in C_{00}(G)$.