Show that on $SL(2,\mathbb{R})/B$ there is no $SL(2,\mathbb{R})$-invariant regular Borel measure.

The following gives a complete answer:

<u>Thm 2.45</u> Let G be l.c.h with left Haar measure μ_G, $H < G$ a closed subgroup with left Haar measure μ_H. Then there is a G-invariant (positive) regular Borel measure on G/H iff

$$\Delta_G\big|_H = \Delta_H .$$

In this case, said regular Borel measure is unique up to positive scalar multiple and there is a unique choice $\mu_{G/H}$

such that Weil's formula holds:

$$\int_G f(g)\, d\mu(g) = \int_{G/H} \int_H f(gh)\, d\mu_H(h)\, d\mu_{G/H}(gH)$$

$\forall f \in C_{00}(G)$.

The proof is based on a key lemma which we proceed to explain. Let μ_H denote a left Haar measure on H. For every $f \in C_{00}(G)$ and $g \in G$, the function $h \mapsto f(gh)$ is in $C_{00}(H)$ and thus

$$T_H f(g) := \int_H f(gh)\, d\mu_H(h)$$

is well defined. A straightforward application of lemma 2.38 (f is right unif. continuous) shows that $g \mapsto T_H f(g)$ is continuous. Since μ_H is left H-invariant we also have $T_H f(gu) = T_H f(g)$ $\forall g \in G$ $\forall u \in H$

and we will henceforth consider $T_H f$ as a function on G/H. Clearly, if $p: G \to G/H$ denotes the projection map, $\operatorname{supp}(T_H f) \subset p(\operatorname{supp} f)$ and thus $T_H f \in C_{00}(G/H)$.

<u>Lemma 2.46</u>: $T_H : C_{00}(G) \to C_{00}(G/H)$ is surjective. In addition, if $F \in C_{00}(G/H)$ is ≥ 0 one can find $f \in C_{00}(G), \geq 0$ with $T_H f = F$.

<u>Proof</u>: We indicate the main point: let $F \in C_{00}(G/H)$; let $K \subset G$ be compact with $p(K) = \operatorname{supp} F$ and $\eta \in C_{00}(G)$ with $0 \leq \eta \leq 1$ and

$\eta|_K \equiv 1$. Now define $f : G \to \mathbb{C}$ by

$$f(g) := \begin{cases} \dfrac{F \circ p(g)\, \eta(g)}{(T_H \eta)(g)} & \text{when } T_H \eta(g) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that f is continuous. It is clearly so on the open subset

$$U_1 := \{ g \in G : T_H \eta(g) \neq 0 \},$$

and also on the open (!) subset $U_2 := G \setminus KH = p^{-1}(G/H \setminus \operatorname{supp} F)$ since it vanishes there. Now observe that if $g \notin U_1$, then $\eta(gh) = 0$ for μ_H-almost every $h \in H$ and hence by continuity for every $h \in H$. But then $g \notin K \cdot H$ (exercise). Thus $U_1 \cup U_2 = G$

and hence f is continuous. Since $\operatorname{supp}(f) \subset \operatorname{supp}(\eta)$ we conclude $f \in C_{00}(G)$.

If $g \in U_1$ then

$$T_H f(g) = \int_H \frac{(F \circ p)(gh)\, \eta(gh)}{T_H \eta(gh)}\, d\mu_H(h)$$

$$= \frac{F \circ p(g)}{T_H \eta(g)} \int_H f(gh)\, d\mu_H(h)$$

$$= F \circ p(g)$$

and if $g \in U_2$, $f(gh) = 0 \quad \forall h \in H$ hence $T_H f(g) = 0 = F \circ p(g)$.

$\boxed{\text{≶}}$

Proof of Thm 2.45.

Given the previous lemma, the proof of Thm 2.45 is astonishingly formal.

(1) Assume that there is a G-invariant regular Borel measure $\mu_{G/H}$ on G/H.

Then for $f \in C_{00}(G)$,

$$I(f) := \int_{G/H} (T_H f)(\dot{g})\, d\mu_{G/H}(\dot{g})$$

clearly defines a left Haar functional on G and in particular $\forall t \in H$

$$I(\rho(t) f) = \Delta_G(t)\, I(f).$$

On the other hand:

$$I(\rho(t) f) = \int_{G/H} T_H(\rho(t) f)(\dot{g})\, d_{G/H}(\dot{g})$$

and $T_H(\rho(t) f)(g) = \int_H f(ght)\, d\mu_H(h)$

$$= \Delta_H(t)\, T_H(f)(g)$$

and thus $\Delta_G|_H = \Delta_H$.

(2) Assume $\Delta_G|_H = \Delta_H$.

Then we claim that $\forall\, f_1, f_2 \in C_{00}(G)$

$$\int_G f_1(g)\, T_H f_2(g)\, d\mu(g) = \int_G f_2(g)\, T_H f_1(g)\, d\mu(g).$$

Indeed:

$$\int_G f_1(g) \int_H f_2(gh)\, d\mu_H(h)\, d\mu_G(g) =$$

$$= \int_H d\mu_H(h) \underbrace{\int_G f_1(g)\, f_2(gh)\, d\mu_G(g)}_{\Delta_G(h) \int_G f_1(gh^{-1})\, f_2(g)\, d\mu_G(g)}$$

$$= \int_G f_2(g) \int_H f_1(gh^{-1}) \underset{\displaystyle\Delta_H(h)}{\underset{\|}{\Delta_G(h)}} \, d\mu_H(h) \, d\mu_G(g)$$

But $\int_H f_1(gh^{-1}) \Delta_H(h) \, d\mu_H(h) =$

$$= \int_H f_1(gh) \, d\mu_H(h)$$

and hence we obtain,

$$\int_G f_2(g) \, T_H f_1(g) \, d\mu_G(g).$$

For $F \in C_0(G/H)$ choose any $f \in C_{00}(G)$ with $T_H f = F$ and "define" $J(F) := \int_G f(g) \, d\mu_G(g)$.

We have to show that $J(F)$ is well defined.

This amounts to show that if $\varphi \in C_{oo}(G)$ satisfies $T_H \varphi = 0$ then $\int_G \varphi(g)\, d\mu_G(g) = 0$. Indeed choose $\psi \in C_{oo}(G)$ such that

$$T_H \psi(\dot{g}) = 1 \quad \forall \dot{g} \in p(\operatorname{supp} \varphi)$$

and compute

$$\int_G \varphi(g)\, d\mu_G(g) = \int_G \varphi(g)\, T_H\psi(g)\, d\mu_G(g)$$

$$= \int_G \psi(g) \underbrace{T_H \varphi(g)}_{\shortparallel \atop 0} d\mu_G(g) = 0.$$

Thus $J : C_{oo}(G/H) \longrightarrow \mathbb{C}$ is well defined; the fact that it is a positive invariant functional is a formal verification.

<u>Example 2.47</u> Recall that if

$G = SL(n, \mathbb{R})$, $K = SO(n, \mathbb{R})$,

$$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$$

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i > 0, \prod_{i=1}^{n} \lambda_i = 1 \right\}$$

then the product map

$$A \times N \times K \longrightarrow G$$

is a homeomorphism. We claim that if μ_A, μ_N, μ_K are left Haar measures then for $f \in C_{00}(G)$,

$$I(f) = \int_A d\mu_A(a) \int_N d\mu_N(n) \int_K d\mu_K(k) \, f(ank)$$

is a left Haar functional.

In deed since G and K are unimodular there is on $G/_K$ a G-invariant regular Borel measure; since $A \cdot N \longrightarrow G/_K$ is an AN-equivariant homeomorphism, this measure corresponds to left Haar measure on AN. It remains to show that for $f \in C_{oo}(AN)$

$$\int_A d\mu_A(a) \int_N d\mu_N(n)\, f(an)$$

is a left Haar functional. Indeed:

$$\int_A d\mu_A(a) \int_N d\mu_N(n)\, f(\underbrace{b \cdot m \cdot an}_{b \cdot a \cdot (\bar{a}^1 m a) n})$$

Now observe $\bar{a}^1 m a \in N$, hence

$$\int_N d\mu_N(n)\, f\big(b \cdot a \cdot (\bar{a}^1 m a) n\big) = \int_N d\mu_N(n) f(b \cdot a \cdot n)$$

and $$\int_A d\mu_A(a) \int_N d\mu_N(n)\, f(b\cdot a\cdot n)$$

$$= \int_A d\mu_A(a) \int_N d\mu_N(n)\, f(a\cdot n).$$