We are closing this chapter with another application of the theory of Haar integral, namely:

<u>Thm 2.48</u> Let $\varphi: G \to H$ be a measurable homomorphism between l.c.h. second countable groups. Then φ is continuous.

We will need two lemmas of general interest:

<u>Lemma 2.49</u> Let G be a top. group, $A \subset W \subset G$ where A is compact and W is open. Then there is a neighborhood $N \ni e$ with $AN \subset W$ (resp. $NA \subset W$).

Proof: For every $x \in A$ let $V_x \ni e$ open with $x . V_x^2 \subset W$. From $A \subset \bigcup_{x \in A} x V_x$ we deduce the existence of $x_1, \dots, x_\ell$ in A with $A \subset \bigcup_{i=1}^{\ell} x_i V_{x_i}$. Let $N := \bigcap_{i=1}^{\ell} V_{x_i}$; then $N \ni e$ is open and:

$$A \cdot N \subset \bigcup_{i=1}^{\ell} x_i V_{x_i} \cdot N \subset \bigcup_{i=1}^{\ell} x V_{x_i}^2 \subset W.$$

$\boxed{\text{B}}$

Lemma 2.50. Let G be a l.c.h. group, μ a left Haar measure and $A \subset G$ compact with $\mu(A) > 0$. Then $AA^{-1} \ni e$ is a neighborhood of e.

Proof: Observe that

$$AA^{-1} = \{x \in G : xA \cap A \neq \emptyset\}.$$

Let $W \supset A$ open with $\mu(W) < 2\mu(A)$

and $N \ni e$ open with $NA \subset W$.

Then $\forall x \in N$:

$$\frac{1}{2}\mu(W) < \mu(A) = \mu(xA) \leq \mu(W)$$

and
$$A \cup xA \subset W$$

which implies that $\mu(A \cap xA) > 0$

and hence $xA \cap A \neq \phi$. This

shows $N \subset AA^{-1}$. $\boxtimes$

<u>Proof of the Theorem.</u>

By restricting to $\overline{\mathrm{Im}\,\varphi}$ the closure

of the image of φ we may assume

$\mathrm{Im}\,\varphi$ is dense in H. Since H is second

countable there is $\{h_n : n \in \mathbb{N}\} \subset \mathrm{Im}\,\varphi$

countable and dense in H.

Let $g_n \in G$ with $\varphi(g_n) = h_n$, $n \geq 1$.

Let $V \ni e_H$ neighborhood of e_H and

$U = U^{-1} \ni e_H$ open with $U^2 \subset V$.

Then $H = \bigcup_{n \geq 1} h_n \cdot U$ and hence $G = \bigcup_{n=1}^{\infty} g_n \varphi^{-1}(U)$

since φ is a homomorphism. Let as

above μ_G be a left Haar measure on

G; then there is n with $\mu_G(g_n \varphi^{-1}(U)) > 0$

and hence $\mu(\varphi^{-1}(U)) > 0$. Let then

$A \subset \varphi^{-1}(U)$ be compact with $\mu(A) > 0$.

Then:

$\varphi^{-1}(V) \supset \varphi^{-1}(U) \varphi^{-1}(U)^{-1} \supset A A^{-1}$ and

by lemma 2.50, $A A^{-1}$ is a neighborhood

of e_G. This shows that φ is continuous

at e_G and hence everywhere. $\boxed{S}$

<u>Example 2.51</u>: The additive groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic: indeed they have the same dimension as $\mathbb{Q}$-vector spaces. It follows then that no isomorphism $\mathbb{R} \longrightarrow \mathbb{R}^2$ is measurable.

To end this chapter we discuss another application of invariant integration in relation with the geometry of numbers.

~~Give~~

We fix once and for all the Lebesgue measure $\mathcal{L}$ on $\mathbb{R}^n$ normalized so that $\mathcal{L}([0,1]^n) = 1$.

Given a lattice $\Lambda < \mathbb{R}^n$ there is a unique $\mathbb{R}^n$-invariant measure $\mathcal{L}_\Lambda$ on $\mathbb{R}^n/\Lambda$ such that:

$$\int_{\mathbb{R}^n} f(x)\, d\mathcal{L}(x) = \int_{\mathbb{R}^n/\Lambda} \Big(\sum_{\lambda \in \Lambda} f(x+\lambda) \Big)\, d\mathcal{L}_\Lambda(x).$$

This formula is valid for $f \in L^1(\mathbb{R}^n)$ and also for $f \geq 0$ measurable.

The fundamental theorem of the geometry of numbers is:

<u>Thm 2.52</u>: Let $\Lambda < \mathbb{R}^n$ be a lattice and $V \subset \mathbb{R}^n$ compact, convex, balanced subset with

$$\mathcal{L}(V) \geq 2^n \mathcal{L}_\Lambda(\mathbb{R}^n/\Lambda).$$

Then $V \cap (\Lambda \setminus \{0\}) \neq \emptyset$.

Balanced means $V = -V$.

<u>Proof:</u>

<u>Claim:</u> $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\Lambda$ is not injective on $\frac{1}{2}V$.

Let's see how the claim implies the theorem. Let $x \neq y$ in $\frac{1}{2}V$ with $p(x) = p(y)$, that is $x - y =: \lambda \in \Lambda$ and $\lambda \neq 0$. Now $x \in \frac{1}{2}V$, $-y \in \frac{1}{2}V$ and hence $\lambda = x - y \in \frac{1}{2}V + \frac{1}{2}V \subset V$.

Now we prove the claim, by contradiction.

Let's assume p is injective on $\frac{1}{2}V$.

Then $p(\frac{1}{2}V) \neq \mathbb{R}^n/\Lambda$. Otherwise

$$\mathbb{R}^n = \bigcup_{\lambda \in \Lambda} (\lambda + \tfrac{1}{2}V)$$

This union is disjoint because p is injective on $\frac{1}{2}V$. In addition this union is locally finite because Λ is discrete and $\frac{1}{2}V$ is compact. Hence $\frac{1}{2}V$ and $\bigcup_{\lambda \neq 0} (\lambda + \frac{1}{2}V)$ are disjoint closed with union $\mathbb{R}^n$, a contradiction.

Thus $p(\frac{1}{2}V) \subsetneq \mathbb{R}^n/\Lambda$.

Since p is injective on $\frac{1}{2}V$ we have:

$$\chi_{p(\frac{1}{2}V)}(x) = \sum_{\lambda \in \Lambda} \chi_{\frac{1}{2}V}(x+\lambda)$$

and hence

$$\mathcal{L}_\Lambda\left(p(\tfrac{1}{2}V)\right) = \int_{\mathbb{R}^n/\Lambda} d\mathcal{L}_\Lambda(x)\, \chi_{p(\frac{1}{2}V)}(x)$$

$$= \int_{\mathbb{R}^n/\Lambda} d\mathcal{L}_\Lambda(x) \sum_{\lambda \in \Lambda} \chi_{\frac{1}{2}V}(x+\lambda) = \int_{\mathbb{R}^n} d\mathcal{L}(x)\, \chi_{\frac{1}{2}V}(x)$$

$$= \mathcal{L}\left(\tfrac{1}{2}V\right) \geqslant \mathcal{L}_\Lambda(\mathbb{R}^n/\Lambda).$$

Hence $\mathcal{L}_\Lambda\left(p\left(\tfrac{1}{2}V\right)\right) = \mathcal{L}_\Lambda(\mathbb{R}^n/\Lambda)$

But $p\left(\tfrac{1}{2}V\right)$ is compact hence closed and the support of $\mathcal{L}_\Lambda$ being $\mathbb{R}^n/\Lambda$, open non-empty subsets have positive $\mathcal{L}_\Lambda$-measure, hence

$$p\left(\tfrac{1}{2}V\right) = \mathbb{R}^n/\Lambda. \quad \text{Contradiction.}$$

$\boxed{3}$

Let $\mathcal{R}^{(1)} = \left\{ \begin{array}{l} \Lambda < \mathbb{R}^n : \Lambda \text{ is a lattice} \\ \text{with } \mathcal{L}_\Lambda(\mathbb{R}^n/\Lambda) = 1 \end{array} \right\}.$

By considering fundamental domains for the Λ-action on $\mathbb{R}^n$, $\mathcal{L}_\Lambda(\mathbb{R}^n/\Lambda) = 1$ iff $\Lambda = \mathbb{Z}f_1 + \dots + \mathbb{Z}f_n$ with

$$\det(f_1, \ldots, f_n) = 1.$$

As a result: $SL(n, \mathbb{R})$ acts transitively on $\mathcal{R}^{(1)}$ and hence

$$SL(n,\mathbb{R})/SL(n,\mathbb{Z}) \longrightarrow \mathcal{R}^{(1)}$$

$$[g] \longmapsto g(\mathbb{Z}^n)$$

is a bijection.

By means of this bijection we transport the topology of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ to $\mathcal{R}^{(1)}$; since $SL(n,\mathbb{R})$ and $SL(n,\mathbb{Z})$ are unimodular there is an $SL(n,\mathbb{R})$-invariant measure on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$; let μ be a fixed $SL(n,\mathbb{R})$-invariant measure on $\mathcal{R}^{(1)}$.

Given a lattice $\Lambda < \mathbb{R}^n$, call $\lambda \in \Lambda$ <u>primitive</u> if $\forall n \geq 2$, $\lambda/n \notin \Lambda$.

Let Λ_{prim} denote the set of primitive vectors in Λ. Given $f \in C_{00}(\mathbb{R}^n)$ define

$$F(\Lambda) := \sum_{\lambda \in \Lambda_{prim}} f(\lambda).$$

Thm 2.53

(1) For $n \geq 1$, $\mu(SL(n,\mathbb{R}) / SL(n,\mathbb{Z})) < +\infty$.

(2) For $n \geq 2$ there is a constant $c_n > 0$ s.t.

$$\int_{\mathcal{R}^{(1)}} F(\Lambda)\, d\mu(\Lambda) = c_n \cdot \int_{\mathbb{R}^n} f(x)\, d\mathcal{L}(x).$$

$$\forall f \in C_{00}(\mathbb{R}^n).$$

Let's indicate how (2) $\Rightarrow$ (1) using Minkowski's theorem. Let

$$V = [-1, 1]^n$$

then $\mathcal{L}(V) = 2^n = \mathcal{L}_\Lambda(\mathbb{R}^n/\Lambda)$ $\forall \Lambda \in \mathcal{R}^{(1)}$. Let $0 \leq f \leq 1$ with $f \in C_{00}(\mathbb{R}^n)$, $f|_V = 1$. By Minkowsky $\forall \Lambda \in \mathcal{R}^{(1)}$:

$$\Lambda \setminus (0) \cap V \neq \emptyset$$

and hence $\Lambda_{prim} \cap V \neq \emptyset$.

This implies

$$F(\Lambda) \geq \sum_{\lambda \in \Lambda_{prim}} \chi_V(\lambda) \geq 1$$

and hence

$$\mu(\mathcal{R}^{(1)}) \leq \int_{\mathcal{R}^{(1)}} F(\Lambda)\, d\mu(\Lambda) = c_n \mathcal{L}(V)$$
$$= 2^n c_n < +\infty .$$