3. Lie groups : basic definitions and general facts.

In this chapter we will introduce the basic objects of Lie theory: Lie groups, their Lie algebras, the exponential map and the adjoint representation. We will prove Cartan's thm that a closed subgroup of a Lie group is a Lie group and discuss the correspondence between Lie (sub)-algebras and Lie (sub)-groups.

3.1. Lie groups and Examples.

We assume the reader has a working knowledge of the basics of differential geometry but we will recall the basic definitions.

<u>Def. 3.1</u> A Lie group is a group G endowed with a structure of smooth manifold such that multiplication $G \times G \to G$ and inversion $G \to G$ are smooth maps.

Recall:

<u>Def. 3.2</u> A topological n-manifold is a second countable Hausdorff space M such that every point in M has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

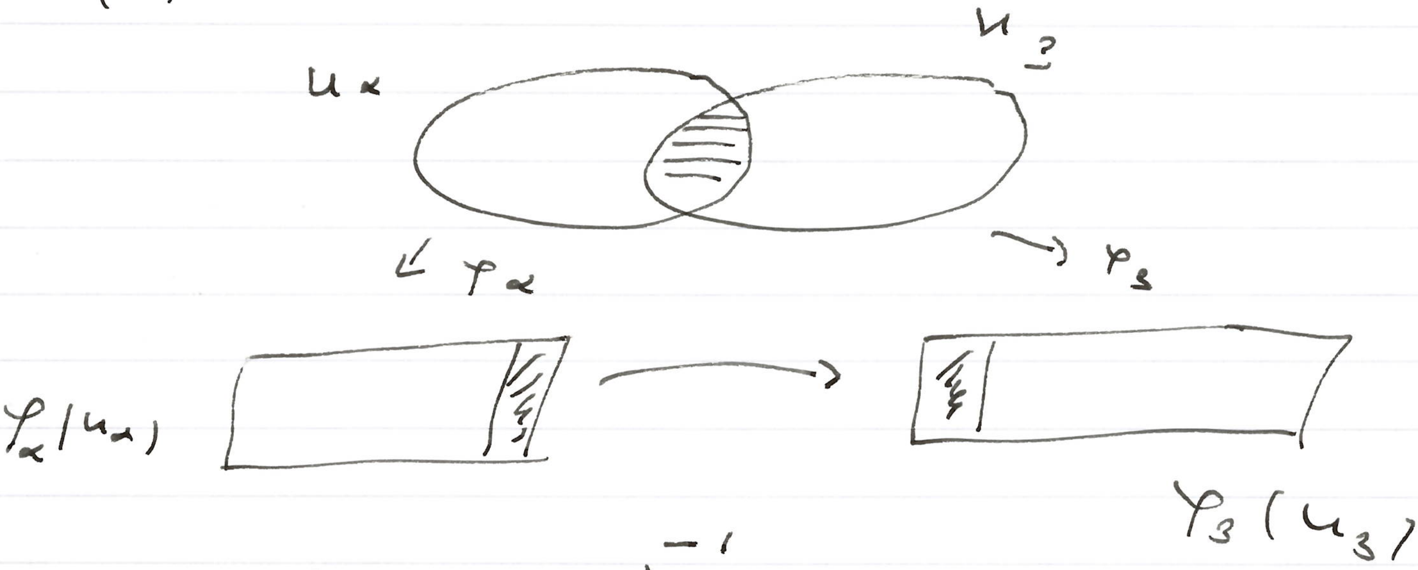
In this context a chart is a pair (U, φ) consisting of an open subset $U \subset M$ and a homeo. $\varphi : U \longrightarrow \underset{\text{open}}{\varphi(U) \subset \mathbb{R}^n}$.

Finally:

<u>Def. 3.3.</u> A smooth structure on a top. n-manifold M is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of charts such that:

(1) $\bigcup_{\alpha \in A} U_\alpha = M$

(2) $\forall \; \alpha, \beta \in A$:



$$\varphi_\beta\Big|_{U_\alpha \cap U_\beta} \left(\varphi_\alpha \Big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} \right)^{-1} :$$

$$\varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

(3) $\mathcal{A}$ is maximal wrt the condition (2).

Any collection of charts satisfying (1) and (2) is called an atlas and one satisfying in addition (3) is called a maximal atlas. One shows that any atlas is contained in a unique maximal one.

Hilbert's fifth problem asked whether there is a purely topological characterization of Lie groups. This has a long history with contributions by many mathematicians (Gleason, Montgomery - Zippin, Yamabe).

One formulation is:

<u>Thm.</u> (1953) A connected topological group that is a topological n-manifold admits a (unique) structure of Lie group.

In contrast Kervaire (1960) constructed a 10 dimensional topological manifold not admitting any smooth structure. For more on Hilbert's 5^{th} problem see:

https://terrytao.files.wordpress.com/2012/03/hilbert-book.pdf

Let's now go through our example list of topological groups in chapter 2 and see which ones can be turned into Lie groups.

<u>Example 3.4</u> (see E. 2.3) A countable discrete group is a 0-dimensional Lie group.

<u>Example 3.5</u> (see E. 2.4, 2.5) $(\mathbb{R}^n, +)$, $(\mathbb{R}^\times, \cdot)$, $(\mathbb{C}^\times, \cdot)$ are (obviously) Lie groups.

<u>Example 3.6.</u> (see E. 2.6)

$GL(n, \mathbb{R})$ is an open subset of $M_{n,n}(\mathbb{R})$ and as such is a smooth n^2-manifold. The matrix product $M_{n,n}(\mathbb{R})^2 \longrightarrow M_{n,n}(\mathbb{R})$ being polynomial is smooth, and so is the inverse $GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$ since $(A^{-1})_{ij} = \dfrac{\det M_{ij}}{\det A}$.

<u>Example 3.7</u> (E 2.7) In general Homeo(X) is not even locally compact, for instance if X is a top. n-manifold with $n \geqslant 1$.

<u>Example 3.8</u> (E 2.8) (X, d) proper metric space then $\mathrm{Iso}(X)$ is l.c.h. which may or may not be a Lie group. For instance if $(X, d) = (\mathbb{R}^n, d_{eucl})$ then $\mathrm{Iso}(X) = O(n, \mathbb{R}) \ltimes \mathbb{R}^n$ which we will show is a Lie group; more generally if (X, d) is a Riemannian manifolds $\mathrm{Iso}(X)$ is a Lie group (Steenrod-Myers 1939).

While (exercise) for $d \geqslant 3$, $\mathrm{Iso}(T_d)$ is not a Lie group.

In order to analyze more example we need some additional tools.

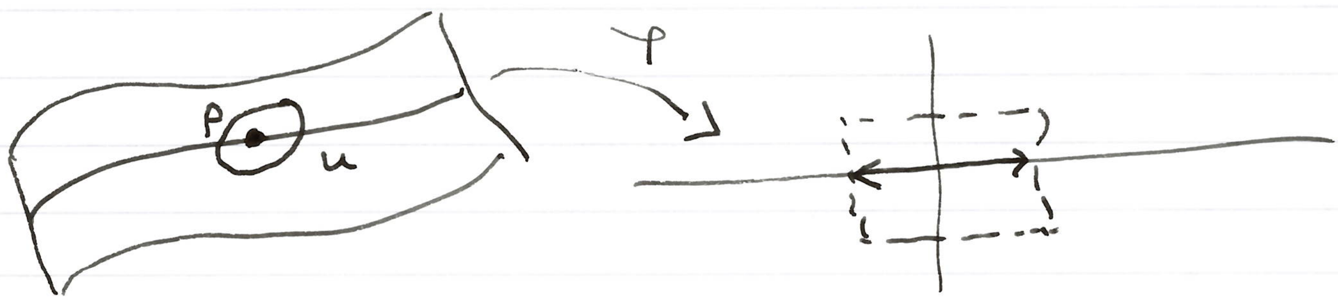
First we need the concept of regular submanifold. Let M be a smooth m-manifold.

<u>Def 3.9</u> A subspace $N \subset M$ is a regular n-submanifold if $\forall p \in N$ there is a chart (U, φ) at p (meaning $p \in U$) such that:

(1) $\varphi(p) = 0$

(2) $\varphi(U) =]-1,1[^m$

(3) $\varphi(N \cap U) = \{x \in]-1,1[^m :$
$x_{n+1} = \dots = x_m = 0\}$



By restricting the charts of Def. 3.9 to N we obtain a smooth n-manifold structure on N.

The usefullness for us of regular submanifolds lies in the following

<u>Thm 3.10</u> (Exercise) Let G be a Lie group and $H < G$ a subgroup which is also a regular submanifold, then H is a Lie group.

<u>Exercise:</u> While a regular submanifold need not be a closed subset, under the hypothesis of Thm 3.10, H is a closed subgroup.

The following consequence of the implicit function theorem gives a powerful way to to construct regular submanifolds.

<u>Thm 3.11</u> Let $f : M \to M'$ be a smooth map of smooth manifolds of dim. resp. m, m'. Assume f has constant rank k on M. Then $\forall q \in f(M)$, $f^{-1}(q)$ is a regular submanifold of M of dim. $m-k$.

Recall that the rank of f at $p \in M$ is the rank of the linear map

$$D_p f : T_p M \to T_{f(p)} M' .$$

The notion of tangent space will be recalled below. In the examples below we will need this Thm applied to M an open subset of some $\mathbb{R}^N$.

Example 3.12

As illustration we consider $SL(n,\mathbb{R})$, $O(n,\mathbb{R})$

and leave $O(p,q)$ as an exercise.

(1) $SL(n,\mathbb{R})$ is a regular (n^2-1)-submanifold

of $GL(n,\mathbb{R})$: let's compute for $A \in GL(n,\mathbb{R})$, and $X \in M_{n,n}(\mathbb{R})$,

$$\left(D_A \det\right)(X) = \frac{d}{dt}\Big|_{t=0} \det(A+tX)$$

Now $\det(A+tX) = \det A \cdot \det\left(I + tA^{-1}X\right)$

hence $\left(D_A \det\right)(X) = \left(\det A\right)\left(D_I \det\right)\left(A^{-1}X\right)$

and hence $\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^\times$ has

constant rank. In fact it has rank 1

since (exercise) $\left(D_I \det\right)(X) = tr(X)$.

(2) $O(n,\mathbb{R})$ is a regular $\frac{n(n-1)}{2}$ - submanifold of $GL(n,\mathbb{R})$:

Consider
$$f : GL(n,\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$$
$$A \longmapsto A^tA$$

and let's compute for $X \in M_{n,n}(\mathbb{R})$:

$$D_A f(X) = \frac{d}{dt}\Big|_{t=0} \underbrace{(A+tX)({}^tA + t\,{}^tX)}_{= A^tA + t(X^tA + A^tX) + t^2 X^tX}$$

$$= X^tA + A^tX$$

Hence: $D_A f(X) = D_I(X^tA)$

and f has constant rank. This rank equals the dimension of $\{X + {}^tX : X \in M_{n,n}(\mathbb{R})\}$ which is $\frac{n(n+1)}{2}$.

3.2. Vector fields and Lie algebras

In this section we recall facts about smooth vector fields and how they lead to an algebraic object called Lie algebra.

Let M be a smooth manifold and $p \in M$. Recall that the ring of germs at p of smooth functions is

$$C^\infty(p) = \{ (U, f) : U \ni p \text{ open}, \underset{\text{smooth}}{f : U \to \mathbb{R}} \} / \sim$$

where $(U_1, f_1) \sim (U_2, f_2)$ if there is $p \in U_3 \subset U_1 \cap U_2$ open with $f_1|_{U_3} = f_2|_{U_3}$.

This has an obvious ring structure; observe that for $f \in C^\infty(p)$, $f(p)$ is well defined.

<u>Def. 3.13</u> A tangent vector at p is a linear form $X_p : C^\infty(p) \longrightarrow \mathbb{R}$ such that $\forall f, g \in C^\infty(p)$

$$X_p(f \cdot g) = f(p)\, X_p(g) + g(p)\, X_p(f),$$

(Leibniz rule)

The set of tangent vectors at p forms a vector space denoted T_pM.

Then one shows (exercise) if (U, φ) is any chart at p with $\varphi(p) = 0$,

$$\mathbb{R}^n \longrightarrow T_pM$$
$$v \longmapsto \left(f \mapsto D_0(f \circ \varphi^{-1})(v) \right)$$

is a vector space isomorphism. In fact this question is purely local and one may assume $M = \mathbb{R}^n$, $p = 0$.

The set of tangent spaces can be organized into a space $TM = \bigsqcup_{p \in M} T_pM$

called the tangent bundle, with a natural smooth structure for which $\pi : TM \longrightarrow M$ $(v,p) \mapsto p$, $v \in T_pM$ is smooth. A smooth vector field is then a smooth section

$$M \longrightarrow TM$$

of π. We will take a different approach and define:

<u>Def. 3.14</u> A vector field on M is a map $X : M \longrightarrow TM$, $p \mapsto X_p$ such that $X_p \in T_pM$ $\forall p \in M$. It is <u>smooth</u> if $\forall f \in C^\infty(M)$,

$$\begin{aligned} M &\longrightarrow \mathbb{R} \\ p &\longmapsto X_p(f) \end{aligned}$$

is smooth.

Local expression of a vector field :

Let (U, φ) be a chart on M. Denoting as usual by $e_1, .., e_n$ the canonical basis of $\mathbb{R}^n$ we get $\forall 1 \leq i \leq n$ a vector field $E^{(i)}$ on U defined by:

$$E^{(i)}_q (f) = D_{\varphi(q)} (f \circ \varphi^{-1})(e_i) \; , \quad q \in U$$
$$f \in C^\infty(U).$$

Since for every $q \in U$,

$$E^{(1)}_q, \ldots, E^{(n)}_q$$

is a basis of $T_q M$ we obtain that if X is any vector field on U, there are uniquely determined functions $g_1, .., g_n$ on U s.t.

$$X_q = \sum_{i=1}^{n} g_i(q) \, E^{(i)}_q$$

We have then that X is smooth $\Leftrightarrow$

$g_1, \ldots, g_n$ are smooth.

It will be important to realize that smooth vector fields on M can be characterized uniquely in terms of the ring $C^\infty(M)$.
To this end we need:

<u>Def. 3.15</u>: Let A be a ~~commutative~~ k-algebra, where k is a field. A derivation of A is an endomorphism

$$\delta : A \to A$$

of the k-vector space A such that

$$\delta(ab) = \delta(a) \cdot b + a \cdot \delta(b) \quad \forall a, b \in A.$$

Let $Der(A)$ = vector space of derivations of A.
~~With this we have then.~~

<u>Prop. 3.16</u> : The map

$$\alpha : \mathrm{Vect}^\infty(M) \to \mathrm{End}(C^\infty(M))$$

defined by $(\alpha X)(f)(p) := X_p(f)$

is an isomorphism into its image

$\mathrm{Der}\, C^\infty(M))$.

<u>Remark 3.17</u> Clearly every $f \in C^\infty(M)$ defines an element in $C^\infty(p)$ $\forall p \in M$, namely the class of (M, f). Conversely $\forall\, U \ni p$ open and $f \in C^\infty(U)$, there is an $F \in C^\infty(M)$ such that (M, F) and (U, f) are equivalent.

This rests on the fact that we can find $g \in C^\infty(M)$ with $\mathrm{supp}(g) \subset U$ and $g = 1$ on a neighborhood of p.

Proof of Prop.

That αX is a derivation of $C^\infty(M)$ follows from Leibniz rule in the definition of tangent vector. Conversely let

$$\delta : C^\infty(M) \to C^\infty(M)$$

be a derivation of $C^\infty(M)$. Fix $p \in M$. Then (exercise) if $f_1, f_2 \in C^\infty(M)$ and f_1, f_2 coincide in a neighborhood U of p then $\delta(f_1)(p) = \delta(f_2)(p)$. $(*)$

[Hint write Leibniz rule for $(f_1 - f_2) \cdot g$ where $g \in C^\infty(M)$ with $\operatorname{supp} g \subset U$].

Now define $X_p : C^\infty(p) \to \mathbb{R}$ by: represent (U, f) by an equivalent (M, F) $F \in C^\infty(M)$ and set $X_p(f) := \delta F(p)$.

because f (*) this X_p is well defined and the derivation property of δ implies $X_p \in T_pM$. $\square$

In general, the composition of two derivations is not a derivation as the simplest example

$$\delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$
$$f \mapsto f'$$

shows.

However:

<u>Lemma 3.18</u>: Let $\delta_1, \delta_2 \in Der(A)$. Then $\delta_1 \cdot \delta_2 - \delta_2 \cdot \delta_1 \in Der(A)$.

<u>Proof</u>: $\cancel{\alpha_1\alpha_2(fg) = \alpha_1[f\,\alpha_2(g) + g\,\alpha_2(f)]}$

$$= \cancel{f \cdot \alpha_1\alpha_2(g) + \alpha_1(f)\alpha_2(g) + \alpha_1(g)\alpha_2(f)}$$
$$\cancel{+ g\,\alpha_1\alpha_2(f)}$$

Proof:

$$\delta_1 \delta_2 (a \cdot b) = \delta_1 \left[\delta_2(a)\, b + a\, \delta_2(b) \right]$$

$$= \delta_1 \delta_2 (a) \cdot b + \delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) + a\, \delta_1\delta_2(b)$$

$$\delta_2 \delta_1 (a \cdot b) = \delta_2 \left[\delta_1(a) \cdot b + a \cdot \delta_1(b) \right]$$

$$= \delta_2 \delta_1(a)\, b + \delta_1(a)\,\delta_2(b) + \delta_2(a)\,\delta_1(b) + a\,\delta_2\delta_1(b)$$

Hence

$$(\delta_1\delta_2 - \delta_2\delta_1)(a \cdot b) =$$

$$= (\delta_1\delta_2 - \delta_2\delta_1)(a) \cdot b + a \cdot (\delta_1\delta_2 - \delta_2\delta_1)(b).$$

~~Thus Given commut k-algebra A~~

~~and $T_1, T_2 \in End(A)$~~

~~Def 3.19 Given $T_1, T_2 \in End(V)$, V a k-~~

~~vector space, define $[T_1, T_2] := T_1T_2 - T_2T_1$.~~

Let us apply this to $Vect^\infty(M)$: given $X, Y \in Vect^\infty(M)$ we conclude from lemma 3.18 that

$$\alpha X \circ \alpha Y - \alpha Y \circ \alpha X \in \operatorname{Der} C^\infty(M)$$

and hence (Prop. 3.16) corresponds to an element in $Vect^\infty(M)$.

<u>Def. 3.19</u> The bracket $[X, Y]$ of two $X, Y \in Vect^\infty(M)$ is the unique element in $Vect^\infty(M)$ with

$$\alpha([X,Y]) = \alpha X \circ \alpha Y - \alpha Y \circ \alpha X.$$

Let us formalise the operation of lemma 3.18 by defining:

<u>Def. 3.20</u> If V is any k-vector space the bracket $[T_1, T_2] \in \operatorname{End}(V)$ of

two endomorphisms T_1, T_2 is :

$$[T_1, T_2] := T_1 T_2 - T_2 T_1 .$$

Then if A is a k-algebra, the bracket operation is a bilinear map on $\mathrm{End}\, A$ preserving $\mathrm{Der}(A)$.

~~Ex~~. The map $\mathrm{End}\, V \times \mathrm{End}\, V \to \mathrm{End}\, V$

$$(T_1, T_2) \longmapsto [T_1, T_2]$$

satisfies:

(1) It is bilinear.

(2) (Antisymmetry) $[T_1, T_2] + [T_2, T_1] = 0$

(3) (Jacobi)

$$[T_1, [T_2, T_3]] + [T_3, [T_1, T_2]] + [T_2, [T_3, T_1]] = 0.$$

The Jacobi identity is a substitute for associativity.

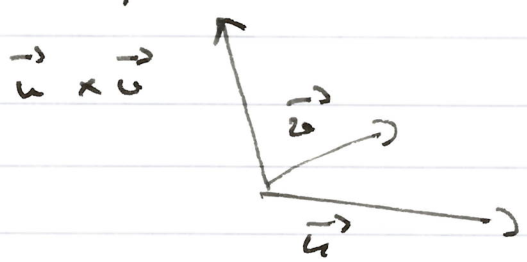
Def 3.21 A Lie algebra over a field k is a k-vector space $\mathfrak{g}$ endowed with a ~~bilinear~~ map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x,y) \mapsto [x,y]$ satisfying (1), (2), (3) above.

Examples 3.22

(1) V a k-vector space : then $\mathrm{End}(V)$ endowed with the bracket is a Lie algebra.

(2) M a smooth manifold, then $\mathrm{Vect}^\infty(M)$ endowed with the bracket is a Lie algebra.

(3) $\mathbb{R}^3$ equipped with the cross product $\times$



<u>Def. 3.23:</u> A k-linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ of k-Lie algebras is a Lie algebra homomorphism if $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ $\forall x, y \in \mathfrak{g}$.

Given now $\varphi : M \to M'$ a smooth map of smooth manifolds, it is clear that in general it will not induce a well defined map $\mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M')$. However it will (not surprisingly) if φ is a diffeomorphism. To understand this we will treat a ~~sing~~ slightly more general situation which will be very useful in the Lie algebra context.

<u>Def. 3.24</u>: We say that $X \in \mathrm{Vect}^\infty(M)$ and $X' \in \mathrm{Vect}(M')$ are φ-related if

$$X'_{\varphi(m)} = D_m\varphi(X_m) \quad \forall m \in M.$$

There is a useful algebraic reformulation.

Let $\varphi^*(f) := f \circ \varphi$, $f \in C^\infty(M')$

then: $\varphi^* : C^\infty(M') \to C^\infty(M')$ is an algebra homomorphism. We have then:

<u>Lemma 3.25</u>: X and X' are φ-related $\Longleftrightarrow$ the diagramm:

$$\begin{tikzcd}
C^\infty(M') \arrow[r, "\varphi^*"] \arrow[d, "\alpha X'"'] & C^\infty(M) \arrow[d, "\alpha X"] \\
C^\infty(M') \arrow[r, "\varphi^*"] & C^\infty(M)
\end{tikzcd}$$

commutes.

Proof: Exercise: it is a reformulation of the definition. $\boxed{\xi}$

Prop. 3.26 If X_i and X_i', $i = 1,2$ are φ-related then $[X_1, X_2]$ and $[X_1', X_2']$ are φ-related.

Proof:

$$\varphi^* \alpha([X_1', X_2']) = \varphi^*\left(\alpha(X_1')\alpha(X_2') - \alpha(X_2')\alpha(X_1')\right)$$

$$= \alpha(X_1)\varphi^*\alpha(X_2') - \alpha(X_2)\varphi^*\alpha(X_1')$$

$$= \alpha(X_1)\alpha(X_2)\varphi^* - \alpha(X_2)\alpha(X_1)\varphi^*$$

$$= \left(\alpha(X_1)\alpha(X_2) - \alpha(X_2)\alpha(X_1)\right)\varphi^*$$

$$= \alpha([X_1, X_2])\varphi^*. \qquad \boxed{\xi}$$

It is clear that if φ is a diffeomorphism then $\varphi^*: C^\infty(M') \longrightarrow C^\infty(M)$ is an algebra isomorphism and hence given $X \in \mathrm{Vect}^\infty(M)$ there is a unique $X' \in \mathrm{Vect}^\infty(M')$ which is φ-related to X, namely:

$$\alpha X' = \varphi^{*-1} \alpha X \, \varphi^*$$

We will denote $X' := \varphi_* X$.

<u>Corollary 3.27</u>: If $\varphi: M \longrightarrow M'$ is a diffeomorphism, then

$$\begin{aligned} \mathrm{Vect}^\infty(M) &\longrightarrow \mathrm{Vect}^\infty(M') \\ X &\longmapsto \varphi_* X \end{aligned}$$

is a Lie algebra isomorphism.