Concerning Def. 3.24 let us recall how the derivative, or tangent map, at $p \in M$ of a smooth map $\varphi : M \to M'$ is defined: let $X_p : C^\infty(p) \to \mathbb{R}$ be a tangent vector and $f \in C^\infty(\varphi(p))$. Let by abuse of notation (U, f) be a representative, $U \ni \varphi(p)$ open. Then

$$(D_p\varphi)(f) := X_p(f \circ \varphi).$$

When working with M being an open subset of a f.d. $\mathbb{R}$-vector space V then we will use a certain number of conventions and identifications.
Let $\Omega \subset V$ be open. We have then the identification (see remark after

Def. 3.13)

$$V \longrightarrow T_v \Omega, \quad v \in \Omega$$
$$w \longmapsto W_v$$

Via $\quad W_v(f) = \frac{d}{dt}\Big|_{t=0} f(v + tw), \quad f \in C^\infty(\Omega)$

If then $L : V \longrightarrow U$ is any linear map:

$$(D_v L)(W_v) = \frac{d}{dt} L(v + tw) = L(w).$$

In particular $\forall \lambda \in V^*$:

$$\underbrace{D_v \lambda (W_v)}_{W_v(\lambda)} = \lambda(w)$$

3.3. The Lie Algebra of a Lie group.

Definition and examples.

Let G be a Lie group and M a smooth manifold.

Def 3.28 A left action of G on M is called smooth if $G \times M \longrightarrow M$ is smooth.

In particular every $g \in G$ gives rise to a diffeomorphism

$$\begin{aligned} L_g : M &\longrightarrow M \\ x &\longmapsto gx \end{aligned}$$

and hence (see Cor. 3.27) to a Lie algebra isomorphism

$$(L_g)_* : \mathrm{Vect}^\infty(M) \longrightarrow \mathrm{Vect}^\infty(M).$$

<u>Def. 3.29</u> $X \in \mathrm{Vect}^\infty(M)$ is G-invariant if $\forall g \in G$: $(L_g)_* X = X$.

The subvector space $\mathrm{Vect}^\infty(M)^G$ of G-invariant vector fields is then a Lie subalgebra of $\mathrm{Vect}^\infty(M)$.

Let now G act on the left on itself:

$$G \times G \to G$$
$$(g, x) \mapsto g \cdot x$$

Then $\mathrm{Vect}_L^\infty(G)^G$ the space of left invariant vector fields is a Lie algebra and

<u>Lemma 3.30</u>
$$\mathrm{Vect}_L^\infty(G)^G \to T_e G$$
$$X \mapsto X_e$$

is a vector space isomorphism.

Proof: We define a map $T_e G \to \mathrm{Vect}^\infty(G)$, $v \mapsto v^L$

as follows:

$$v^L_g := D_e L_g(v).$$

The fact that $v^L \in \mathrm{Vect}^\infty_L(G)^G$ follows from the chain rule.

Also $v^L_e = v$ since $L_e = \mathrm{id}_G$.

Finally if $X \in \mathrm{Vect}^\infty_L(G)^G$, we have in particular:

$$X_g = (D_e L_g)(X_e)$$

and hence $X = (X_e)^L$.

∎

~~Now we proceed to identify explicitely the Lie algebra of $GL(n,\mathbb{R})$, which is a key step in computing the Lie~~

<u>Def. 3.31</u>: The Lie algebra $\mathfrak{g}$ of a Lie group G is the vector space $\mathfrak{g} = T_e G$ endowed with the bracket:

$$[v, w] := [v^L, w^L]_e \qquad \forall\, v, w \in T_e G.$$

We now proceed to identify explicitely the Lie Algebra of $GL(n, \mathbb{R})$. Recall that since $GL(n, \mathbb{R}) \subset M_{n,n}(\mathbb{R})$ is open we have the identification:

$$M_{n,n}(\mathbb{R}) \longrightarrow T_I GL(n, \mathbb{R})$$
$$A \longmapsto A_I$$

Let us denote $\mathfrak{gl}(n, \mathbb{R})$ the Lie algebra of $GL(n, \mathbb{R})$ and for convenience $\tilde{A} = (A_I)^L$ the left invariant vector field corresponding to A_I. Then:

<u>Proposition 3.32</u> The map

$$M_{n,n}(\mathbb{R}) \longrightarrow gl(n, \mathbb{R})$$
$$A \longmapsto A_I$$

induces an isomorphism between the Lie algebra $M_{n,n}(\mathbb{R})$, with matrix bracket and the Lie algebra $gl(n,\mathbb{R})$.

In other words:

$$[\tilde{A}, \tilde{B}] = \widetilde{[A, B]} \quad \forall A, B \in M_{n,n}(\mathbb{R}).$$

<u>Proof</u>: Since $[\tilde{A}, \tilde{B}]$ and $\widetilde{[A,B]}$ both are left invariant vector fields, it suffices to show that

$$[\tilde{A}, \tilde{B}]_I = \widetilde{[A,B]}_I .$$

But (see p. 3.30) two tangent vectors at I coincide iff their evaluation on all $\lambda \in M_{n,n}(\mathbb{R})^*$ do.

Thus we want to show:

$$[\tilde{A}, \tilde{B}]_I(\lambda) = \widetilde{[A,B]}_I(\lambda)$$

$$\forall \lambda \in M_{n,n}(\mathbb{R})^*.$$

But $\widetilde{[A,B]}_I = [A, B]_I$ and hence

$$\widetilde{[A,B]}_I(\lambda) = [A,B]_I(\lambda) = \lambda(AB - BA)$$

$$= \lambda(AB) - \lambda(BA).$$

On the other hand:

$$[\tilde{A}, \tilde{B}]_I(\lambda) = (\tilde{A}\tilde{B} - \tilde{B}\tilde{A})(\lambda)(I)$$

and we proceed to show:

$$\tilde{A}\tilde{B}(\lambda)(I) = \lambda(AB).$$

Now $\widetilde{A}\widetilde{B}(\lambda)(I) = A_I(\widetilde{B}(\lambda))$

$= A_I(g \mapsto \widetilde{B}_g(\lambda))$

$= A_I(g \mapsto D_I L_g(B_I)(\lambda))$

But $D_I L_g(B_I)(\lambda) = B_I(h \mapsto \lambda(gh))$

But $h \mapsto \lambda(gh)$ is the restriction to $GL(n,\mathbb{R})$ of a linear form on $M_{n,n}(\mathbb{R})$

hence $B_I(h \mapsto \lambda(gh)) = \lambda(gB)$.

Again for the same reason:

$A_I(g \mapsto \lambda(gB)) = \lambda(AB)$

and we are done. $\boxed{\Sigma}$

Now we turn to the natural question of whether a smooth homomorphisms of Lie groups induces a Lie algebra homomorphism. This will in particular allow us to determine the Lie algebras of certain subgroups of $GL(n, \mathbb{R})$.

<u>Prop. 3.33</u> Let $\varphi : G \longrightarrow H$ be a smooth homomorphism of Lie groups and $\mathfrak{g} = T_e G$, $\mathfrak{h} = T_e H$ their Lie algebras. Then $\quad D_e\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

<u>Proof:</u>

Let $v \in T_e G$, $v^L \in \mathrm{Vect}^\infty(G)^G$ the corresponding left inv. vector field,

$w := D_e\varphi(v) \in T_eH$ and $w^L \in \mathrm{Vect}^\infty(H)$.

Claim: v^L and w^L are φ-related

(see Def. 3.24).

Indeed:

$$\underline{w^L_{\varphi(g)}} = D_e L_{\varphi(g)}(w) = D_e L_{\varphi(g)} D_e\varphi(v)$$

$$= D_e(L_{\varphi(g)} \circ \varphi)(v)$$

since φ is a homomorphism,

$$L_{\varphi(g)} \circ \varphi = \varphi \circ L_g ,$$

hence

$$= D_e(\varphi \circ L_g)(v) = D_g\varphi(D_e L_g(v))$$

$$= \underline{D_g\varphi(v^L_g)} .$$

Thus if $v_1, v_2 \in T_eG$, and

$w_i := D_e\varphi(v_i)$ then since v_i^L and w_i^L

are φ-related, it follows from prop 3.26 that $[v_1^L, v_2^L]$ and $[w_1^L, w_2^L]$ are φ-related. Hence:

$$D_e\varphi([v_1, v_2]) = D_e\varphi\left([v_1^L, v_2^L]_e\right) = [w_1^L, w_2^L]_e$$

$$= [w_1, w_2] = [D_e\varphi(w_1), D_e\varphi(w_2)].$$

$\boxed{\S}$

<u>Corollary 3.34.</u> Let G be a Lie group and $H < G$ a subgroup which is also a regular submanifold. Then the inclusion $H \to G$ realizes $\mathfrak{h} = T_e H$ as Lie subalgebra of $\mathfrak{g} = T_e G$.

Examples 3.35

(1) The Lie algebra of $SL(n,\mathbb{R})$ is

$$\mathfrak{sl}(n,\mathbb{R}) = \{ X \in M_{n,n}(\mathbb{R}) : \operatorname{tr} X = 0 \}$$

(Ex. 3.12 (1))

Indeed, indeed $SL(n,\mathbb{R}) = \det^{-1}(1)$

and $\det : GL(n,\mathbb{R}) \to \mathbb{R}^{*}$ has

constant rank with $(D_I \det)(X) = \operatorname{tr} X$.

(2) The Lie algebra of $O(n,\mathbb{R})$ is

$$\mathfrak{o}(n,\mathbb{R}) = \{ X \in M_{n,n}(\mathbb{R}) : X + {}^{t}X = 0 \}$$

for the same reason (see Ex. 3.12 (2)).

(3) $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ is

obviously a subgroup and a regular

submanifold of $GL(n,\mathbb{R})$ and its Lie algebra is:

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}.$$

(4) $$A = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}^* \right\}$$

then $$\mathfrak{a} = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}.$$

Observe that $[\ ,\ ]$ vanishes on $\mathfrak{a}$.

Exercises 3.36

(1) Compute the Lie algebra of $O(p,q)$, $SO(p,q)$, $p+q=n$;

(2) Realize $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$ as Lie groups and compute their Lie algebras.

<u>Example 3.37</u>: Let G, H be Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$. Then the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with the bracket:

$$[(x_1, y_1), (x_2, y_2)] = \left([x_1, x_2]_{\mathfrak{g}}, [y_1, y_2]_{\mathfrak{h}}\right).$$

(left as an exercise).

At this point we are left with lots of questions: we have constructed a functor

$$\mathrm{Lie} : \mathrm{Lie\,Grps} \longrightarrow \mathrm{Lie\,Alg},$$

and the fundamental question is how much information we loose by going from Lie groups to Lie algebras. Loosely here are some informal answers:

(1) Every Lie algebra is the Lie algebra of a Lie group.

(2) (Faithfulness).
If G is a Lie group and F a finite group with discrete topology then $G \times F$ and G have the same Lie algebra. Even if G is connected, its Lie algebra doesn't determine G:

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 / \mathbb{Z}^2$$

is a covering map and it is easy to see that it induces an isomorphism between the invariant vector fields.

In fact if G_1, G_2 are connected Lie groups then any isomorphism

$$\mathfrak{g}_1 \to \mathfrak{g}_2$$

comes from an isomorphism

$$\tilde{G}_1 \to \tilde{G}_2 .$$

(3) It is desirable that the category of Lie groups is closed under certain natural operations like taking the center $Z(G)$ of a Lie group G, or G^0 the connected comp. of identity etc.

In this direction we will show a striking thm. of Cartan, namely: if $H < G$ is a closed subgroup of a Lie group G then it is a regular submanifold, hence a Lie group.

3.4. The exponential map

The exponential map is, among other things, a computational tool that links a Lie group to its Lie algebra. It is obtained from the simple observation that a left invariant vector field generates a one parameter group of diffeomorphisms of a special type.
In the special case of $GL(n,\mathbb{R})$ the Lie group exponential will turn out to be the matrix exponential which we proceed to study first.

Choose any norm $\| \ \|$ on $\mathbb{R}^n$ and endow $M_{n,n}(\mathbb{R})$ with the operator norm

$$\|A\| := \sup_{\|v\| \leq 1} \|Av\|$$

which clearly satisfies

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|.$$

We have then

<u>Prop. 3.38</u>:

(1) The series $\displaystyle\sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges uniformly on balls of finite radius in $M_{n,n}(\mathbb{R})$ to a smooth map called Exp.

(2) For all A, B with $[A, B] = 0$,

$$\mathrm{Exp}(A+B) = \mathrm{Exp}(A) \cdot \mathrm{Exp}(B)$$

in particular Exp takes values in $GL(n, \mathbb{R})$.

(3) For all $A \in M_{n,n}(\mathbb{R})$, the map $\varphi : \mathbb{R} \longrightarrow GL(n,\mathbb{R})$ $t \mapsto \mathrm{Exp}(tA)$ is a smooth homomorphism with

$$\dot{\varphi}(0) = A$$

(4) Any smooth homomorphism

$$\varphi : \mathbb{R} \longrightarrow GL(n,\mathbb{R})$$

is of the form $\varphi(t) = \mathrm{Exp}(t\dot{\varphi}(0))$.

Proof:

(1) For all A with $\|A\| \leq R$ and $N \geq 1$ we have
$$\Big\| \sum_{n=N}^{\infty} \frac{A^n}{n!} \Big\| \leq \sum_{n=N}^{\infty} \frac{R^n}{n!} .$$

Which shows uniform convergence on balls of radius R.

Concerning partial derivatives:

we have

$$\frac{\partial}{\partial x_{ij}} X^n = \sum_{k_1+k_2=n-1} X^{k_1} E_{ij} X^{k_2} \qquad (*)$$

which implies

$$\left\| \frac{\partial}{\partial x_{ij}} X^n \right\| \leq n \cdot \| X^{n-1} \|$$

and applying $(*)$ interatively shows

that if $\dfrac{\partial^I}{\partial x_I}$ is a partial derivative

of order k:

$$\left\| \frac{\partial^I}{\partial x_I} X^n \right\| \leq n(n-1)\cdots(n-k+1) \, \| X^{n-k} \|$$

for $n \geq k$.

This shows that $\displaystyle\sum_{n=0}^{\infty} \frac{\partial^I}{\partial x_I} \frac{X^n}{n!}$

converges uniformly on balls of finite radius and hence Exp is smooth. In fact real analytic.

(2) This follows from the fact that if $AB = BA$ then

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}.$$

Also:

$$I = \mathrm{Exp}(0) = \mathrm{Exp}(A)\,\mathrm{Exp}(-A)$$

which shows Exp takes values in $GL(n,\mathbb{R})$.

(3) Clear from (1) and (2).

(4) $\varphi : \mathbb{R} \to GL(n,\mathbb{R})$ then

$$\dot{\varphi}(t) = \frac{d}{ds}\Big|_{s=0} \varphi(t+s) = \varphi(t)\,\dot{\varphi}(0).$$

By uniqueness of solutions of ODE we get $\varphi(t) = \operatorname{Exp}(t \cdot \dot{\varphi}(0))$. $\square$

Now we turn to the construction of the exponential map for a general Lie group. For this we recall an existence result concerning integral curves of smooth vector fields.

<u>Definition 3.33</u> An integral curve of a smooth vector field X on M is a smooth map $\gamma : I \longrightarrow M$ with

$$\dot{\gamma}(t) = X_{\gamma(t)} \quad \forall t \in I.$$

Here: $I \subset \mathbb{R}$ is an open interval and $\dot{\gamma}(t) := (D_t \gamma)(1)$, $1 \in T_t\mathbb{R}$.

The fundamental existence and uniqueness theorem for first order ordinary differential equations in $\mathbb{R}^n$ implies: (See Boothby IV.4)

<u>Thm. 3.40</u> Let $X \in \mathrm{Vect}^\infty(M)$. For every $m \in M$ there exist $a(m)$, $b(m)$ in $\mathbb{R} \cup \{\pm\infty\}$ and a smooth curve

$$\gamma_m : (a(m), b(m)) \longrightarrow M$$

s.t.

(1) $0 \in (a(m), b(m))$ and $\gamma(0) = m$.

(2) γ_m is an integral curve of X

(3) if $\mu : (c, d) \longrightarrow M$ is a smooth curve satisfying (1) and (2) then $(c, d) \subset (a(m), b(m))$ and $\gamma|_{(c,d)} = \mu$.

Def. 3.41. The vector field $X \in \mathrm{Vect}^\infty(M)$ is complete if $\forall m \in M$,

$$(a(m), b(m)) = \mathbb{R}$$

that is, the integral curves given by Thm 3.40 are defined on $\mathbb{R}$.

Corollary 3.42 Let $X \in \mathrm{Vect}^\infty(M)$ be complete. Then the map

$$\overset{X}{\Phi} : \mathbb{R} \times M \longrightarrow M$$
$$(t, m) \longmapsto \gamma_m(t)$$

is a smooth map satisfying

$$\overset{X}{\Phi}(t_1 + t_2, m) = \overset{X}{\Phi}(t_1, \Phi(t_2, m))$$
$$\forall t_1, t_2 \in \mathbb{R}, \ \forall m \in M.$$

One calls $t \mapsto \Phi(t, ?)$ a 1-parameter group of diffeomorphisms.

<u>Proof</u>: The map $t \mapsto \gamma_m(t_2+t)$ is an integral curve through $\gamma_m(t_2)$: by uniqueness we get

$$\gamma_m(t_2+t) = \gamma_{\gamma_m(t_2)}(t) .$$

Reformulating this in terms of Φ^X gives the result. ☒

Let $X, Y \in \mathrm{Vect}^\infty(M)$, which we assume complete for simplicity. Then the <u>Lie derivative</u> of Y wrt X at $p \in M$ is:

$$(L_X Y)_p =$$

$$= \lim_{t \to 0} \frac{1}{t} \left[\left(D_{\Phi^X(t,p)} \Phi^X_{-t} \right) \left(Y_{\Phi^X(t,p)} \right) - Y_p \right]$$

which leads to the following interpretation of the bracket of two vector fields:

$$L_X Y = [X,Y] \quad \left(\text{Boothby, Thm 7.8} \atop \text{chap. IV}\right)$$

This is used in the proof of

<u>Prop. 3.43</u>

Let X, Y be complete smooth vector fields on M. Then: $\Phi^X_t \cdot \Phi^Y_s = \Phi^Y_s \Phi^X_t$ $\forall\, t, s \in \mathbb{R}$ iff $[X,Y] = 0$.

Now we come back to Lie groups and invariant vector fields:

<u>Prop. 3.44</u> Let G be a Lie group.

(1) Left invariant vector fields are complete.

(2) For every $v \in T_e G$, let $v^L \in \mathrm{Vect}^\infty(G)$