which leads to the following interpretation of the bracket of two vector fields:

$$L_X Y = [X, Y] \quad \left(\text{Boothby, Thm 7.8} \atop \text{chap. IV}\right)$$

This is used in the proof of

Prop. 3.43

Let X, Y be complete smooth vector fields on M. Then: $\Phi_t^X \cdot \Phi_s^Y = \Phi_s^Y \Phi_t^X$ $\forall t, s \in \mathbb{R}$ iff $[X, Y] = 0$.

Now we come back to Lie groups and invariant vector fields:

Prop. 3.44 Let G be a Lie group.

(1) Left invariant vector fields are complete.

(2) For every $v \in T_eG$, let $v^L \in \mathrm{Vect}^\infty(G)^G$

be the corresponding left invariant vector field and $\gamma_{(v)} : \mathbb{R} \to G$ the integral curve of v^L through e. Then $\gamma_{(v)}$ is a smooth homomorphism.

(3) The one parameter group of diffeomorphisms

$$\Phi^{v^L} : \mathbb{R} \times G \to G$$

is given by $\Phi^{v^L}(t, g) = g\,\gamma_{(v)}(t)$.

<u>Proof:</u>

(1) Let $\gamma_e : (a(e), b(e)) \to G$ be the integral curve of v^L through e given by Thm. 30.40. We claim that

$$(G) \quad \gamma_g(t) := g\,\gamma_e(t) \quad , \quad t \in (a(e), b(e))$$

is an integral curve of v^L through g.

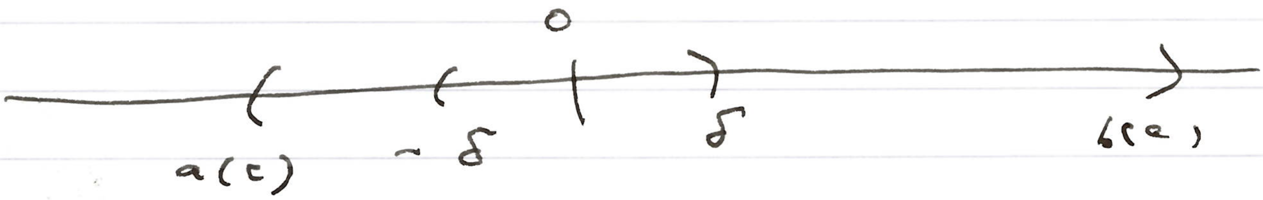
Indeed:

$$\dot{\gamma}_g(t) = D_{\gamma_e(t)} L_g \left(\dot{\gamma}_e(t) \right) = D_{\gamma_e(t)} L_g \left(v^L_{\gamma_e(t)} \right)$$

$$= v^L_{g\gamma_e(t)}$$

the last equality using left invariance of v^L.

Let now $\delta > 0$ such that



and define:

$$\gamma(t) := \begin{cases} \gamma_e(t) & t \in (a(e), b(e)) \\ \\ \gamma_e(\delta) \cdot \gamma_e(t-\delta), & \\ & t \in (a(e)+\delta, b(e)+\delta) \end{cases}$$

This is well defined since by the above (c)

$$t \mapsto \gamma_e(t) \quad \text{and} \quad t \mapsto \gamma_e(\delta)\gamma_e(t-\delta)$$

are both integral curves of v^L through $\gamma_e(\delta)$, and hence coincide on any common interval of definition.

It follows that γ is an integral curve through e defined on

$$(a(e), b(e) + \delta)$$

which by Thm 3.40 (3) implies $b(e) = +\infty$.

A similar argument gives $a(e) = -\infty$.

Thus by (c) this implies that v^L is complete. Thus Φ^{v^L} is defined on $\mathbb{R} \times G$ and it follows from (G) that

$$\Phi^{v^L}(t, g) = g \cdot \Phi^{v^L}(t, e).$$

This shows (1) and (3).

Concerning (2) we have since Φ^{G^L} is a 1-parameter group of diffeomorphisms:

$$\Phi(t_1+t_2, g) = \Phi(t_1, \Phi(t_2, g))$$

$$= \Phi(t_2, g)\, \Phi(t_1, e)$$

and hence since $t_1 + t_2 = t_2 + t_1$:

$$\Phi(t_1+t_2, e) = \Phi(t_1, e)\, \Phi(t_2, e)$$

which concludes the proof of (2) since

$$\varphi_{(v)}(t) = \Phi^{G^L}(t, e). \qquad \square$$

In this context it is natural to introduce the following Lie group concept:

<u>Def. 3.45</u> A one parameter group in G is a smooth homomorphism $\mathbb{R} \to G$.

We have seen that a tangent vector $v \in T_e G$ leads via the corresponding left invariant vector field to a one parameter group $\varphi_{(v)} : \mathbb{R} \longrightarrow G$.

In fact the converse holds:

<u>Corollary 3.46.</u> If $\varphi : \mathbb{R} \longrightarrow G$ is a one parameter group then

$$\varphi = \varphi_{(v)} \quad \text{where } v = \dot{\varphi}(0).$$

<u>Proof:</u> Let $v = \dot{\varphi}(0) \in T_e G$ and v^L the corresponding left invariant vector field. We have

$$\dot{\varphi}(t) = \frac{d}{ds}\Big|_{s=0} \varphi(t+s) = \frac{d}{ds}\Big|_{s=0} \varphi(t)\varphi(s)$$

$$= D_e L_{\varphi(t)} (\underbrace{\dot{\varphi}(0)}_{v}) = v^L_{\varphi(t)} .$$

hence φ is an integral curve through e of v^L hence $\varphi = \varphi_{(v)}$. $\blacksquare$

Now we are in a position to define the exponential map in general:

<u>Def. 3.47</u> Let G be a Lie group with Lie Algebra $\mathfrak{g}$. The exponential map

$$\exp_G : \mathfrak{g} \longrightarrow G$$

is defined by $\exp_G(v) = \varphi_{(v)}(1)$,

where $\varphi_{(v)}$ is the integral curve of v^L through e.

<u>Corollary 3.48</u>: We have:

(1) $\exp_G(t \cdot v) = \varphi_{(v)}(t) \quad \forall t \in \mathbb{R}, \forall v \in \mathfrak{g}$.

(2) If $v, w \in \mathfrak{g}$ satisfy $[v,w] = 0$

Then $\exp_G(v+w) = \exp_G(v)\exp_G(w)$.

The proof of (2) requires a lemma:

<u>Lemma 3.49</u> Let $m: G\times G \to G$ be the product map. Then under the identification of $T_{(e,e)}(G\times G)$ with $T_eG\times T_eG$ we have
$$D_{(e,e)}m(v,w) = v+w.$$

<u>Proof</u>: Since $D_{(e,e)}m : T_eG\times T_eG \to T_eG$ is linear we have:

$$D_{(e,e)}m(v,w) = D_{(e,e)}m(v,0) + D_{(e,e)}m(0,w).$$

Consider now:
$$\begin{array}{ccccc} G & \xrightarrow{i_1} & G\times G & \xrightarrow{m} & G \\ g & \longmapsto & (g,e) & \longrightarrow & g\cdot e \end{array}$$

Then $m\circ i_1 = id_G$ and hence

$$D_{(e,e)} m \underbrace{D_e i_1(v)}_{(v,0)} = 0 \qquad \forall v \in T_e G$$

$\boxed{\varepsilon}$

Proof of Corollary 3.48

(1) By definition $\exp_G(t\cdot v) = \varphi_{(t\cdot v)}(1)$.

Consider now $\psi(s) := \varphi_{(v)}(t\cdot s)$.

Then ψ is a one parameter group with $\dot\psi(0) = t\cdot\dot\varphi_{(v)}(0) = t\cdot v$ and hence by Cor. 3.46, $\psi = \varphi_{(t\cdot v)}$ which implies

$$\varphi_{(v)}(t\cdot s) = \varphi_{(t\cdot v)}(s) \qquad \forall s$$

and hence $\varphi_{(t\cdot v)}(1) = \varphi_{(v)}(t)$.

that is $\exp_G(t\cdot v) = \varphi_{(v)}(t)$.

$\boxed{\varepsilon}$

(2) If $[v, w] = 0$ we have (Prop. 3.43)

$$\Phi_t^{v^L} \Phi_s^{w^L} = \Phi_s^{w^L} \cdot \Phi_t^{v^L} \quad \forall t, s$$

and hence by Prop. 3.44 (3):

$$\varphi_{(v)}(t)\, \varphi_{(w)}(s) = \varphi_{(w)}(s)\, \varphi_{(v)}(t) \quad \forall t, s.$$

This implies that:

$$\psi(t) := \varphi_{(v)}(t)\, \varphi_{(w)}(t) \quad , t \in \mathbb{R}$$

is a 1-parameter group in G with

$$\dot{\psi}(0) = D_{(e,e)} m \left(\dot{\varphi}_{(v)}(0), \dot{\varphi}_{(w)}(0) \right)$$

$$= D_{(e,e)} m \,(v, w)$$

$$= v + w \quad \text{by lemma 3.47.}$$

Hence $\psi(t) = \varphi_{(v+w)}(t)$ which implies

$$\exp_G(v + w) = \exp_G(v) \exp_G(w). \quad \boxed{\text{III}}$$

The characterization of 1-parameter groups in terms of the exponential map leads to the following striking

<u>Prop. 3.~~48~~ 50</u> Let $\varphi : G \to H$ be a smooth homomorphism. Then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ T_eG & \xrightarrow[D_e\varphi]{} & T_eH \\ \| & & \| \\ \mathfrak{g} & & \mathfrak{h} \end{array}$$

commutes.

<u>Proof</u>: The map $\psi : \mathbb{R} \to H$

$$t \mapsto \varphi(\exp_G(t \cdot \omega))$$

is a 1-parameter group in H with

$\dot{\psi}(0) = D_e\varphi(\omega)$. Hence by

Cor. 3.48 (1) and Cor. 3.46 we have

$$\psi(t) = \exp_H t \cdot D_e\varphi(u)$$

which proves the prop. $\boxed{3}$

<u>Exercise 3.51:</u> Use Prop. 3.38 (3) to show that $\exp_{GL(n,\mathbb{R})}(t \cdot A) = \operatorname{Exp}(t \cdot A)$

$t \in \mathbb{R}$, $A \in \mathfrak{gl}(n, \mathbb{R}) = M_{n,n}(\mathbb{R})$.

The exponential map gives a prefered chart at e:

<u>Corollary 3.52</u> Let G be a Lie group with Lie Algebra $\mathfrak{g}$.

(1) $D_0 \exp_G = \mathrm{Id}_{\mathfrak{g}}$

(2) There is $0 \in U \subset \mathfrak{g}$ open such that $\exp_G(U) \subset G$ is open and

$$\exp_G : \mathcal{V} \to \exp_G(\mathcal{V})$$

is a diffeomorphism.

<u>Proof:</u> For every $X \in \mathfrak{g}$, $\left.\frac{d}{dt}\right|_{t=0} \exp_G(t \cdot X) = X$ which shows (1). Then the inverse function thm. implies (2). $\square$

<u>Example 3.53</u>

(1) (E. Cartan) If K is a compact connected Lie group, $\exp_K : \mathfrak{k} \to K$ is surjective. The proof goes via constructing a biinvariant Riemannian metric on K and showing that the Riemannian exponential coincides with $\exp_K$; surjectivity follows then from Hopf-Rinow.

It is an exercise to show that

$$\mathrm{Exp} : \mathfrak{u}(n) \longrightarrow U(n)$$

is surjective. Combine the fact that every $A \in U(n)$ is diagonalisable with the formula $g\,\mathrm{Exp}(X)g^{-1} = \mathrm{Exp}(gXg^{-1})$ valid for all $X \in M_{n,n}(\mathbb{C})$, $g \in GL(n,\mathbb{C})$.

(2) A similar argument using the Jordan normal form implies that

$$\mathrm{Exp} : \mathfrak{gl}(n,\mathbb{C}) \longrightarrow GL(n,\mathbb{C})$$

is surjective.

(3) Let $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} : x \in \mathbb{R} \right\}$$

Since $X^n = 0 \quad \forall X \in \mathfrak{n}$ we have

$$\operatorname{Exp} X = Id + X + \frac{X^2}{2!} + \cdots + \frac{X^{n-1}}{(n-1)!}$$

Here Exp is not only surjective, it is a polynomial diffeomorphism $\mathfrak{n} \longrightarrow N$.

One can find an explicit inverse to Exp_N by remembering that for $x \in \mathbb{R}_{>0}$

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k} .$$

Now if $g \in N$, then $g - Id$ is nilpotent, $(g - Id)^n = 0$ and one verifies that

$\operatorname{Ln} : N \longrightarrow \mathfrak{n}$,

$$\operatorname{Ln}(g) = (g - Id) - \frac{(g-Id)^2}{2} + \cdots + \frac{(-1)^{n}(g-Id)^{n-1}}{n}$$

is an inverse to $\operatorname{Exp} : \mathfrak{n} \longrightarrow N$.

(4) $\mathrm{Exp} : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow SL(2,\mathbb{R})$ is <u>not</u> surjective. Indeed since

$$\left(\mathrm{Exp}\left(\tfrac{X}{2}\right)\right)^2 = \mathrm{Exp}(X)$$

every matrix in the image of Exp is a square. But $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not.

Observe however that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One of the first consequences of the properties of the exponential map is a structure theorem for connected abelian Lie groups.

<u>Def. 3.54</u> A Lie algebra $\mathfrak{g}$ is abelian if $[\,,\,] \equiv 0$.

We have then

<u>Prop. 3.5.5</u> :

(1) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G is abelian iff $\mathfrak{g}$ is abelian.

(2) Let G be a connected abelian Lie group. Then $\exp_G : \mathfrak{g} \longrightarrow G$ is a smooth surjective homomorphism, its kernel $\Gamma := \operatorname{Ker} \exp_G$ is a discrete subgroup of $\mathfrak{g}$ and $\exp_G$ induces an isomorphism of Lie groups

$$\mathfrak{g}/_{\Gamma} \simeq G.$$

Proof: We show (1) and (2) at the same time.

Assume G connected abelian; then Prop. 3.44 (2) implies that for all $v, w \in \mathfrak{g}$:

$$\Phi^{v^L}_t \cdot \Phi^{w^L}_s = \Phi^{w^L}_s \cdot \Phi^{v^L}_t$$

and hence Prop. 3.43 implies

$$[v^L, w^L] = 0$$

that is $[v, w] = 0$, $\forall v, w \in \mathfrak{g}$.

Assume now $\mathfrak{g}$ abelian and G connected. Cor 3.48 (2) implies that $\exp_G : \mathfrak{g} \to G$ is a (smooth) homomorphism; by Cor. 52 (2), $\exp_G(\mathfrak{g})$ is an open subgroup of G hence closed; since G is connected we obtain $\exp_G(\mathfrak{g}) = G$ and G is abelian. Let $U \ni 0$ open in $\mathfrak{g}$ s.t. $\exp_G : U \to \exp_G(U) \subset G$ is a open diffeo. Then $\Gamma \cap U = (0)$ and Γ is a discrete subgroup. It is then an exercise in diff. geom. to show that the induced group isomorphism

$$\mathfrak{g}/_\Gamma \longrightarrow G$$

is a diffeo. $\boxed{S}$

Exercise 3.56 Let V be a finite dimensional real vector space and $\Gamma < V$ a discrete subgroup. Show that there are $\gamma_1, \dots, \gamma_r$ in Γ, linearly independent in V s.t. $\Gamma = \mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_r$.

<u>Exercise 3.57</u>: Show that every connected abelian Lie group G is isomorphic as Lie group to

$$\mathbb{T}^a \times \mathbb{R}^{n-a}$$

where $n = \dim G$, $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$.

<u>Exercise 3.58</u>: Let G be a Lie group. Show that there is an open neighborhood of e which does not contain any non-trivial subgroup of G.

3.5 Cartan's Theorem on closed subgroups.

We now come to

Thm. 3.59 Let G be a Lie group and $H < G$ a closed subgroup. Then H is a regular submanifold of G and hence a Lie group.

Given $\mathfrak{g} = A \oplus B$ a vector space direct sum decomposition we get special coordinates in a neighb. of e:

Lemma 3.60 Let π_A, π_B be the projections of $\mathfrak{g}$ onto A, B. Then the map

$$\varphi : \mathfrak{g} \longrightarrow G$$
$$\xi \longmapsto \exp_G \pi_A(\xi) \exp_G \pi_B(\xi)$$

has derivative $\mathrm{Id}_{\mathfrak{g}}$ at $\xi = 0$.

Proof: This is a computation using lemma 3.49:

$$\varphi(\xi) = m\left(\exp_G \pi_A(\xi), \exp_G \pi_B(\xi)\right)$$

so

$$D_0\varphi(X) = D_{(e,e)} m \left(\underbrace{D_0 \exp_G \pi_A(X)}_{\pi_A(X)}, \underbrace{D_0 \exp_G \pi_B(X)}_{\pi_B(X)} \right)$$

$$= \pi_A(X) + \pi_B(X) = X. \quad \boxed{\xi}$$

In particular φ provides us with a chart near e.

Proof of Cartan's Thm.

Let $H < G$ be a closed subgroup and fix a norm $\| \ \|$ on $\mathfrak{g}$.

Let $S^1 = \{ v \in \mathfrak{g} : \|v\| = 1 \}$

and $\pi : \mathfrak{g} \setminus (0) \longrightarrow S^1$

$$v \longmapsto \frac{v}{\|v\|}$$

the projection map.

Let

$W = \{0\} \cup \Big\{ \xi \in \mathfrak{g} \setminus (0)$: there exists $(v_n)_{n \geq 1}$ in $\mathfrak{g} \setminus (0)$ such that

$\exp_G(v_n) \in H \ \forall n \geq 1$, $\lim_{n \to \infty} v_n = 0$

and $\lim_{n \to \infty} \pi(v_n) = \pi(\xi) \Big\}$.

This subset W is clearly a cone and it is our candidate to be a "tangent space at e to H".

We now proceed in three steps:

(1) $\underline{\exp_G(W) \subset H}$:

We have $e = \exp_G(0) \in H$. Let $\xi \in W$, $\xi \neq 0$ and (v_n) as in the definition of W:

$$\lim_{n\to\infty} \frac{v_n}{\|v_n\|} = \frac{\xi}{\|\xi\|}.$$

Hence $\displaystyle\lim_{n\to\infty} \frac{\|\xi\|}{\|v_n\|} \cdot v_n = \xi$.

Let $a_n := \left[\frac{\|\xi\|}{\|v_n\|}\right]$ the integer part of $\frac{\|\xi\|}{\|v_n\|}$. We claim that

$$\lim_{n\to\infty} a_n \cdot v_n = \xi.$$

Indeed:

$$\left\| \frac{\|\xi\|}{\|v_n\|} v_n - a_n v_n \right\| = \left| \frac{\|\xi\|}{\|v_n\|} - a_n \right| \|v_n\| \leq \|v_n\|$$

and $\displaystyle\lim_{n\to\infty} \|v_n\| = 0$.

Thus

$$\exp_G(\xi) = \lim_{n\to\infty} \exp_G(a_n v_n)$$

$$= \lim_{n\to\infty} \left(\exp_G(v_n)\right)^{a_n} \in H.$$

(2) <u>W is a vector subspace of $\mathfrak{g}$.</u>

We already observed that W is a cone. Let $\xi, \eta \in W$: we may assume that ξ, η and $\xi+\eta$ are all non zero.

We have since $\exp_G(W) \subset H$ and W is a cone:

$$\exp_G t\xi \cdot \exp_G t\eta \in H \quad \forall t.$$

Since $\exp_G$ gives a local chart at e there is $I_\delta = (-\delta, \delta)$, $\delta > 0$, and a smooth curve $u : I_\delta \longrightarrow \mathfrak{g}$ with $u(0) = 0$

such that $\exp_G u(t) = \exp_G t\xi \cdot \exp_G t\cdot\eta$

$t \in I_\delta$.

From lemma 3.49 we get:

$$\left.\frac{d}{dt}\right|_{t=0} \exp_G u(t) = \left(D_0 \exp\right)(\dot{u}(0))$$

$$= D_{(e,e)}\, m\,(\xi,\eta) = \xi+\eta$$

hence $\dot{u}(0) = \xi+\eta$ since $D_0 \exp = \mathrm{id}_{\mathfrak{g}}$.

Define $v_n := u(1/n)$, $n \geq 1/\delta$

since $\lim u(1/n) = u(0) = 0$ we have $\lim \| v_n \| = 0$.

Next:

$$n\, v_n = n\, u(1/n) = \frac{u(1/n) - u(0)}{1/n - 0} \longrightarrow \xi+\eta$$

which implies $v_n \neq 0$ for n large

and $\lim\limits_{n\to\infty} \pi(v_n) = \pi(\xi+\eta)$.

Thus $\xi + \eta \in W$.

(3) There is an open $\mathcal{U} \ni 0$ in $\mathfrak{g}$ and a diffeomorphism $\Phi : \mathcal{U} \to \Phi(\mathcal{U})$ where $\Phi(\mathcal{U}) \subset G$ is open and $\Phi(0) = e$ s.t.

$$\Phi(\mathcal{U} \cap W) = \Phi(\mathcal{U}) \cap H.$$

Let W' be a vector space complement to W:

$$\mathfrak{g} = W \oplus W'.$$

From lemma 3.60 we deduce that there is $\mathcal{U} \ni 0$ open in $\mathfrak{g}$ such that the map

$$\Phi : \mathfrak{g} \longrightarrow G$$
$$u \longmapsto \exp_G \pi_W(u) \exp_G \pi_{W'}(u)$$

restricted to $\mathcal{U}$ is a diffeomorphism with the open set $\Phi(\mathcal{U}) \subset G$.

Evidently $\Phi(0) = e$ and clearly

$$\Phi(\mathcal{V} \cap W) \subset \Phi(\mathcal{V}) \cap H.$$

We want to show the existence of an open subset $0 \in \mathcal{U}' \subset \mathcal{V}$ such that

$$\Phi(\mathcal{U}' \cap W) = \Phi(\mathcal{U}') \cap H.$$

We proceed by contradiction. Thus assume there is a sequence $(\mathcal{U}_n)_{n \geq 1}$ of open subsets of $\mathcal{V}$ with:

(1) $0 \in \mathcal{U}_n$, $\forall n \geq 1$

(2) $\mathcal{U}_n \subset \mathcal{U}_{n-1}$, $\forall n \geq 2$

(3) $\Phi(\mathcal{U}_n \cap W) \subsetneq \Phi(\mathcal{U}_n) \cap H$, $\forall n \geq 1$

(4) $\bigcap_{n \geq 1} \mathcal{U}_n = \{0\}$.

From (3) we get for every $n \geq 1$ a vector $v_n + v_n' \in \mathcal{U}_n$ such that $v_n \in W$, $v_n' \in W'$, $v_n' \neq 0$ and $\exp_G(v_n)\exp_G(v_n') \in H$, hence

$$\exp_G(v_n') \in H \quad \forall n \geq 1.$$

Passing to a subsequence we may assume

$$\xi := \lim_{n \to \infty} \pi(v_n') \in S^1$$

exists. Since $v_n + v_n' \in \mathcal{U}_n$ and $\bigcap_{n \geq 1} \mathcal{U}_n = \{0\}$ we get $\lim_{n \to \infty} \|v_n'\| = 0$.

Thus $\xi \in W$; on the other hand:

$$\xi = \lim_{n \to \infty} \frac{v_n'}{\|v_n'\|} \in W'$$

and hence $\xi \in W \cap W' = (0)$, a contradiction.

$\boxed{\xi}$