From the proof of Cartan's theorem we deduce

<u>Cor 3.61</u>. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H < G$ a closed subgroup. Then its Lie algebra is given by
$$\mathrm{Lie}(H) = \{ X \in \mathfrak{g} : \exp_G(tX) \in H \ \forall t \in \mathbb{R} \}.$$

This Corollary together with the explicit relationship between a smooth homomorphism and its derivative has many consequences.

<u>Cor. 3.62</u>: Let $\pi : G_1 \to G_2$ be a smooth homomorphism with derivative $d\pi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$.

then $\mathrm{Lie}(\mathrm{Ker}\,\pi) = \mathrm{Ker}\, d\pi$.

<u>Proof</u>: By Cor. 3.61,

$$\mathrm{Lie}(\mathrm{Ker}\,\pi) = \{X \in \mathfrak{g} : \pi(\exp_G(tX)) = e \quad \forall t \in \mathbb{R}\}$$

By Prop. 3.50:

$$\pi(\exp_G tX) = \exp_H(t\, d\pi(X))$$

hence the above equals

$$= \{X \in \mathfrak{g} : \exp_H(t \cdot d\pi(X)) = e \quad \forall t \in \mathbb{R}\}$$

$$= \{X \in \mathfrak{g} : d\pi(X) = 0\}$$

$$= \mathrm{Ker}(d\pi). \qquad \boxed{\xi}$$

Now that we know that a closed subgroup H of a Lie group G is a regular submanifold we also expect some additional structure on the homogeneous space G/H. This is indeed the case; in fact let V be a complement of $\mathfrak{h}$ in $\mathfrak{g}$: $\mathfrak{g} = V \oplus \mathfrak{h}$. Fix as usual a norm $\| \ \|$ on $\mathfrak{g}$ and consider $D_\varepsilon := \exp_G \{ X \in V : \|X\| < \varepsilon \}$

Then one can show (see Bröcker, Dieck p. 34) that the mapping

$$m : D_\varepsilon \times H \longrightarrow G \ , \ (g, h) \longmapsto g \cdot h$$

is, for ε sufficiently small, an open embedding.

This leads to:

<u>Thm 3.63</u> The operation of H on G by right multiplication defines an H-principal bundle with total space G, structure group H, base space G/H, and projection $\pi: G \longrightarrow G/H$, $g \mapsto gH$. In particular G/H is a smooth manifold, and π is a submersion.

<u>Cor. 3.64</u> If in addition $H \triangleleft G$ then the group G/H with the above differentiable structure is a Lie group.

It is now time to introduce

<u>Def. 3.65</u>. An ideal $\mathfrak{a}$ in a k-Lie algebra $\mathfrak{g}$ is a vector subspace such that

$$[X,Y] \in \mathfrak{n} \quad \forall X \in \mathfrak{g}, \forall Y \in \mathfrak{n}.$$

The Lie bracket on $\mathfrak{g}$ descends then to $\mathfrak{g}/\mathfrak{n}$ to define a Lie algebra structure s.t. $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ is a Lie algebra homomorphism with $\operatorname{Ker} \pi = \mathfrak{n}$. Conversely if $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, $\operatorname{Ker} \varphi$ is an ideal in $\mathfrak{g}_1$.

<u>Cor. 3.66</u> In the situation of Cor. 3.64 the projection $\pi : G \to G/H$ induces a surjective Lie algebra homomorphism

$$d\pi : \mathfrak{g} \to \operatorname{Lie}(G/H)$$

with kernel $\mathfrak{h} = \operatorname{Lie}(H)$.

3.6. The adjoint representation.

We begin by setting some general terminology.

Def. 3.67 (1) A representation of a Lie group G in a real or complex vector space V is a smooth homomorphism

$$\pi : G \longrightarrow GL(V).$$

(2) A representation of a Lie algebra in a real or complex vector space V is a Lie algebra hom.

$$\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

We have seen that a representation $\pi : G \to GL(V)$ leads to a representation $d\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of its Lie algebra.

Remembering that the Lie group exponential of $GL(V)$ is the matrix exponential we have:

$$\pi(\exp_G(X)) = \mathrm{Exp}(d\pi(X)) \quad \forall X \in \mathfrak{g}.$$

Let $W \subset V$ be a vector subspace and $v \in V$; let

$$\mathrm{Stab}(W) = \{ g \in G : \pi(g) W \subset W \}$$

$$\mathrm{Stab}(v) = \{ g \in G : \pi(g) v = v \}.$$

These are closed subgroups and

<u>Prop. 3.68</u>: $\mathrm{Lie}\,\mathrm{Stab}(W) = \{ X \in \mathfrak{g} : d\pi(X) W \subset W \}$

$\mathrm{Lie}\,\mathrm{Stab}(v) = \{ X \in \mathfrak{g} : d\pi(X) v = 0 \}.$

Evidently we have for $g \in G$:

$$\pi(g)\,W \subset W \iff \pi(g)\,W = W.$$

<u>Proof</u>:

(1) $$\mathrm{Lie\,Stab}(W) = \{X \in \mathfrak{g} : \pi(\exp_G tX)\,W \subset W\}$$

$$= \{X \in \mathfrak{g} : \mathrm{Exp}\, t\, d\pi(X)\, W \subset W\}$$

It is then an exercise that for $A \in \mathfrak{gl}(V)$

$$\mathrm{Exp}\, tA\,(W) \subset W \quad \forall t \in \mathbb{R}$$

$$\iff \quad A(W) \subset W.$$

(2) $$\mathrm{Lie\,Stab}(w) = \{X \in \mathfrak{g} : \mathrm{Exp}\, t\, d\pi(X)\, w = w \quad \forall t \in \mathbb{R}\}$$

and a similar exercise shows:

$$\mathrm{Exp}\, tA\,(w) = w \quad \forall t \in \mathbb{R}$$

$$\iff \quad A(w) = 0.$$

∎

We turn now to the definition and study of the adjoint representation. Beside the exponential map it is the second most fundamental tool in the study of Lie groups.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For $g \in G$, $\operatorname{int}(g) : G \longrightarrow G$, $x \mapsto g x g^{-1}$ is a smooth automorphism and we denote by

$$\operatorname{Ad}(g) := D_e \operatorname{int}(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

its derivative. We clearly have $\operatorname{Ad}(e) = \operatorname{Id}_{\mathfrak{g}}$ and the chain rule gives $\operatorname{Ad}(g_1 g_2) = \operatorname{Ad}(g_1)\operatorname{Ad}(g_2)$, $\forall g_1, g_2 \in G$.

<u>Exercise</u>: $\begin{array}{l} G \longrightarrow GL(\mathfrak{g}) \\ g \longmapsto \operatorname{Ad}(g) \end{array}$ is smooth.

From Prop. 3.50 we deduce the fundamental relation: $\forall g \in G$, $\forall t \in \mathbb{R}$, $\forall X \in \mathfrak{g}$:

$$(\text{F.R.}) \qquad g \exp_G(tX) g^{-1} = \exp_G(t\, \mathrm{Ad}(g) X)$$

With this we can compute Ad for $G = GL(n, \mathbb{R})$: for $g \in GL(n, \mathbb{R})$, $X \in \mathfrak{gl}(n, \mathbb{R})$, $t \in \mathbb{R}$:

$$\sum_{k=0}^{\infty} \frac{t^k (\mathrm{Ad}(g) X)^k}{k!} = g \left(\sum_{k=0}^{\infty} \frac{t^k X^k}{k!} \right) g^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{t^k (g X g^{-1})^k}{k!}$$

which upon comparing coefficients gives:

$$\mathrm{Ad}(g) X = g X g^{-1} .$$

~~Given any Lie~~.

In analogy with the G-action by conjugation on G a Lie algebra $\mathfrak{g}$ acts on itself by the bracket:

$$\begin{aligned} ad: \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\longmapsto [X, \cdot] \end{aligned}$$

it is the adjoint representation of $\mathfrak{g}$ and the fact that it is a Lie algebra homomorphism is equivalent to the Jacobi identity.

<u>Theorem 3.69</u> Let G be a Lie group $Ad: G \to GL(\mathfrak{g})$ its adjoint representation and $ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint representation of its Lie algebra.

Then: $D_e Ad(X) = ad(X) \quad \forall X \in \mathfrak{g}$.

<u>Proof</u>: We will obtain the above identity by retranslating everything in terms of left invariant vector fields.

Let $h \in G$ and $Y \in \mathfrak{g}$:

then $Ad(h)Y = D_e(\mathrm{int}(h))Y$ leads to: $\forall f \in C^\infty(G)$, $\forall g \in G$:

$$[Ad(h)Y]^L(f)(g) = \frac{d}{ds}\Big|_{s=0} f(g\, h \exp_G(sY) h^{-1})$$

and hence $\forall t \in \mathbb{R}$, $X \in \mathfrak{g}$:

$$(*)\ \big(Ad[\exp tX]\,Y\big)^L(f)(g) =$$

$$= \frac{d}{ds}\Big|_{s=0} f\big(g \exp(tX) \exp(sY) \exp(-tX)\big).$$

Now we use Prop. 3.50 and the explicit form for the exponential map of $GL(g)$.

$$\mathrm{Ad}(\exp tX) = \mathrm{Exp}\left(t\, D_e \mathrm{Ad}(X)\right)$$

$$= \sum_{k=0}^{\infty} \frac{t^k \left(D_e \mathrm{Ad}(X)\right)^k}{k!}$$

hence: $\forall f \in C^\infty(G),\ \forall g \in G$:

$$(**)\quad \left(\mathrm{Ad}(\exp tX)(Y)\right)^L (f)(g) =$$

$$= \sum_{k=0}^{\infty} \frac{t^k \left[\left(D_e \mathrm{Ad}(X)\right)^k (Y)\right]^L (f)(g)}{k!}$$

From $(**)$ we get:

$$\frac{d}{dt}\Big|_{t=0} \left[\mathrm{Ad}(\exp tX)(Y)\right]^L (f)(g) =$$

$$= \left[D_e \mathrm{Ad}(X)(Y)\right]^L (f)(g).$$

And from $(*)$ we get then:

$$\left[D_e \mathrm{Ad}(X)(Y)\right]^L (f)(g) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f\left(g \exp(tX) \exp(sY) \cdot \exp(-tX)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f\left(g \exp(tX) \exp(sY)\right)$$

$$- \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f\left(g \exp(sY) \exp(tX)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} (Y^L f)(g \exp tX) - \frac{d}{ds}\Big|_{s=0} (X^L f)(g \exp sY)$$

$$= X^L(Y^L f)(g) - Y^L(X^L f)(g)$$

and hence

$$\left[D_e \mathrm{Ad}(X)(Y)\right]^L (f)(g) = [X^L, Y^L](f)(g).$$

$\square$

As a first application we show:

<u>Corollary 3.70</u>: Let $N < G$ be a closed subgroup with Lie algebra $\mathfrak{n} < \mathfrak{g}$.

Then (1) If N is normal, $\mathfrak{n} \triangleleft \mathfrak{g}$ is an ideal.

(2) Conversely if G and N are connected and $\mathfrak{n}$ of $\mathfrak{g}$ is an ideal, N is normal.

<u>Proof</u>:

(1) Uses Prop. 3.68 and is left as an exercise.

(2) If $ad(X)$ leaves $\mathfrak{n}$ invariant then so does $\operatorname{Exp} ad(X)$ and hence $Ad(\exp X) = \operatorname{Exp}(ad(X))$ leaves $\mathfrak{n}$ invariant.

~~Thus $\forall X \in \mathfrak{g}, \forall Y \in \mathfrak{n}$,~~

~~$int(\exp X)(\exp Y) = \exp[Ad(\exp X)(Y)] \in N$.~~

Thus $\forall X \in \mathfrak{g}, \forall Y \in \mathfrak{n}$:

$Ad(\exp X)(Y) \in \mathfrak{n}$ which implies

$$\exp\left[Ad(\exp X)(Y)\right] \in N.$$

But the latter equals

$int(\exp X)(\exp Y) \in N$, $\forall X \in \mathfrak{g}$, $\forall Y \in \mathfrak{n}$.

Thus the subgroup of G generated by $\exp \mathfrak{g}$ leaves the subgroup of N generated by $\exp \mathfrak{n}$ invariant. Since both G and N are connected this implies the statement. $\boxtimes$

4. Structure Theory.

The aim is to decompose a connected Lie group into "simple" pieces. This will be done via its Lie algebra. Various properties of Lie algebras then appear, like solvability, nilpotency and semisimplicity. The aim is then to translate this properties at the level of the corresponding Lie groups.

4.1. Solvable Lie groups and Lie Algebras.

Solvability is a group theoretic property coming from Galois theory.

Def. 4.1 A group G is solvable if there is a sequence of subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = \{e\}$$

with G_i / G_{i+1} abelian $\forall\ 0 \leq i \leq r-1$.

In order to analyze solvability, the derived series of a group G will play an important role.

Let $G^{(1)}$: the subgroup of G generated by the set $\{[x, y] : x, y \in G\}$ of commutators.

Def. 4.2. The derived series of a group G is defined inductively by

$$G^{(i)} := [G^{(i-1)}, G^{(i-1)}], \quad i \geq 2.$$

We record the following elementary facts:

Lemma 4.3

(1) Let $\pi : G \to H$ be a homomorphism, then $\pi(G^{(i)}) = \pi(G)^{(i)}$, $i \geq 1$.

(2) Let $N \triangleleft G$. Then G/N is abelian $\Leftrightarrow N \supset G^{(1)}$.

<u>Proof</u> (1) Clear, since $\pi([x,y]) = [\pi(x), \pi(y)]$.

(2) Let $\pi : G \to G/_N$ be the canonical projection. Then from (1) we get:

$$[G/_N, G/_N] = [\pi(G), \pi(G)] = \pi([G,G]).$$

Hence $G/_N$ is abelian $\Leftrightarrow [G,G] \subset \operatorname{Ker}\pi = N$.

$\boxed{B}$

With this we have:

<u>Lemma 4.4.</u> G is solvable $\Leftrightarrow \exists\, r \geq 1$ with $G^{(r)} = (e)$.

<u>Proof</u>:

($\Leftarrow$) Clearly $G^{(i)} \triangleleft G^{(i-1)}$ and by lemma 4.3 (2), $G^{(i-1)}/G^{(i)}$ is abelian.

($\Rightarrow$) Let $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = (e)$

as in the def. of solvability. We have $G_i \supset G^{(i)}$: by recurrence, $G/_{G_1}$ is abelian hence $G_1 \supset G^{(1)}$. If $G_{i-1} \supset G^{(i-1)}$, we have since G_{i-1}/G_i is abelian

$$G_i \supset (G_{i-1})^{(1)} \supset \left(G^{(i-1)}\right)^{(1)} = G^{(i)} . \ \square$$

<u>Def. 4.5</u> For G solvable, the solubility length of G is

$$sol(G) := \min\{r \geq 1 : G^{(r)} = (e)\}.$$

From lemma 4.2 (1) we get

<u>Lemma 4.6.</u> Let $N \trianglelefteq G$ and $N \supset G^{(s-1)}$ then $sol(G/N) \leq r-1$.

<u>Proof:</u> for $\pi : G \to G/N$ we have:

$$(G/N)^{(s-1)} = \pi(G)^{(s-1)} = \pi\left(G^{(s-1)}\right) = (e). \ \square$$

Now we show that for Hausdorff top. groups that are solvable the subgroups from the definition can be chosen with additional properties.

<u>Theorem 4.7</u> Let G be a solvable topological Hausdorff group. Then $\exists$ a sequence

$$G = G_0 \rhd G_1 \rhd G_2 \cdots \rhd G_r = \{e\}$$

with G_{i-1}/G_i abelian, $1 \leq i \leq r-1$ and G_i closed. If G is connected the G_i's can be taken connected.

<u>Exercise 4.8</u> Let G be a top. group, $N \lhd G$ closed subgroup. If N and G/N are connected then so is G.

<u>Lemma 4.9</u> If G is a connected top. group then $G^{(i)}$ is connected $\forall i \geq 1$.

<u>Proof</u>: The set $V := \{[x,y] : x, y \in G\}$ is connected and so are all the $V^n = V \cdots V$. But $G^{(1)} = \bigcup_{n \geq 1} V^n$ and $V^n \ni e$ $\forall n \geq 1$.

$\boxed{3}$

<u>Proof</u> by rec. on $sol(G) := r$.

If $r=1$, $G^{(1)} = (e)$ and (e) is closed!.

Let $r \geqslant 2$: $G^{(r-1)}$ is abelian and normal in G, hence $\overline{G^{(r-1)}}$ is normal and abelian. Also $sol\left(G/\overline{G^{(r-1)}} \right) \leq r-1$ and $G/\overline{G^{(r-1)}}$ is a Hausdorff top. group.

Now ~~take~~ consider $\pi : G \to G/\overline{G^{(r-1)}}$ and apply the recurrence hypothesis to obtain $G/\overline{G^{(r-1)}} = H_0 \triangleright H_1 \cdots \triangleright H_l = (e)$ with H_i closed and H_{i-1}/H_i abelian.

Then

$$G = \pi^{-1}(H_0) \triangleright \pi^{-1}(H_1) \cdots \triangleright \pi^{-1}(H_l) = \overline{G^{(r-1)}}$$

does the job. If in addition G is connected then $\overline{G^{(r-1)}}$ is connected, one may take the H_i's connected and then $\pi^{-1}(H_i)$ are connected. $\boxed{\checkmark}$

This together with the classification of abelian connected Lie groups leads to

Corollary 4.10. Let G be a connected solvable Lie group. Then there is a sequence

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = (e)$$

where G_i is closed connected and G_{i-1}/G_i is isomorphic either to $\mathbb{R}$ or $\mathbb{T}$ $1 \leq i \leq r-1$.

Proof: Let $G = H_0 \triangleright H_1 \triangleright \cdots H_\ell = (e)$ (see Thm 4.7) with H_i closed connected and H_{i-1}/H_i abelian. By exercise 3.57 $H_{i-1}/H_i \simeq \mathbb{T}^{a_i} \times \mathbb{R}^{b_i}$. Now take inverse image in H_{i-1} of the sequence

$$(e) \subset \mathbb{T} \subset \cdots \subset \mathbb{T}^{a_i} \subset \mathbb{T}^{a_i} \times \mathbb{R} \subset \cdots \subset \mathbb{T}^{a_i} \times \mathbb{R}^{b_i}$$

$\boxed{3}$

<u>Example 4.11:</u>

$$G = \left\{ \begin{pmatrix} \boxed{*} & \cdots & * \\ & \ddots & \vdots \\ 0 & & \boxed{*} \end{pmatrix} : \begin{array}{l} * \in \mathbb{R} \\ \boxed{*} \in \mathbb{R}^\times \end{array} \right\} \subset GL(n,\mathbb{R})$$

Then

$$G^{(1)} = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

$$G^{(2)} = \left\{ \begin{pmatrix} 1 & 0 & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix} \right\}$$

and $G^{(n)} = \{ \mathrm{Id} \}$.

This is the prototype of solvable Lie group.
This leads us to the first fundamental theorem of Lie

<u>Thm 4.12</u> Let G be a connected Lie group that is solvable as a group and $\rho : G \to GL(V)$ a representation into a $\underline{\mathbb{C}}$ – vector space V.
Then there is a basis of V such that

$\rho(g)$ is upper triangular $\forall g \in G$.

<u>Def. 4.13</u> Let G be a Lie group and $\rho: G \to GL(V)$ a complex representation. A weight of G in V is a homomorphism

$$\chi : G \to \mathbb{C}^\times$$

such that $V_\chi := \left\{ v \in V : \begin{array}{l} \rho(g)v = \chi(g)v \\ \forall g \in G \end{array} \right\} \neq \{0\}$.

Then V_χ is the weight space and any $v \in V_\chi \setminus \{0\}$ is a weight vector.

<u>Rem.</u> a weight is automatically smooth.

In fact the central point from which Lie's thm follows is:

<u>Thm. 4.14</u>, Let G be connected Lie group that is solvable and $\rho : G \to GL(V)$ a complex rep. Then G has a weight in V.

This will rely on the following lemma:

<u>Lemma 4.15</u> Let G be a connected Lie group $\rho : G \to GL(V)$ a complex rep. of G, $H \triangleleft G$ and $\chi : H \to \mathbb{C}^\times$ a weight of H in $\rho|_H : H \to GL(V)$. Then V_χ is $\rho(G)$-invariant.

<u>Proof</u>: Given $g \in G$, $h \in H$, $v \in V_\chi$ we have:

$$\rho(h)\rho(g)v = \rho(g)\rho(g^{-1}hg)v = \chi(g^{-1}hg)\rho(g)v.$$

Now $\chi(g^{-1}hg) \in \mathrm{Spec}(\rho(h)) \subset \mathbb{C}^\times$

Thus we get a continuous map

$$\begin{aligned} G &\longrightarrow \mathrm{Spec}(\rho(h)) \\ g &\longmapsto \chi(g^{-1}hg) \end{aligned}$$

Since G is connected and $\mathrm{Spec}(\rho(h))$ finite, this map is constant. Hence $\chi(g^{-1}hg) = \chi(h)$ $\forall g \in G$, $\forall h \in H$ which implies $\rho(g)V_\chi \subset V_\chi$.

$\boxed{3}$

Proof of Thm. 4.14.

By recurrence on $\dim G$.

If $\dim G = 1$, $\dim \mathfrak{g} = 1$, so $\mathfrak{g} = \mathbb{R} \cdot X$.

Let $v_0 \neq 0$ in V be an eigenvector of $d\rho(X)$; then $d\rho(Y)\mathbb{C}v \subset \mathbb{C}v$ $\forall Y \in \mathfrak{g}$ and since G is connected this implies by Prop. 3.68 that $\rho(G)\mathbb{C}v = \mathbb{C}v$.

But then $\rho(g)v = \chi(g)v$ and χ is a weight.

Let $\dim G \geq 2$ and let $H \triangleleft G$ closed connected normal with $G/H \simeq \mathbb{T}$ or $\mathbb{R}$.

By recurrence H has a weight $\chi : H \to \mathbb{C}^*$ in V and by lemma 4.15, V_χ is $\rho(G)$-invariant.

From $\rho(h)v = \chi(h)v$, $v \in V_\chi$, $h \in H$

we deduce $d\rho_e(X)v = (D_e\chi)(X)v$ $\forall X \in \mathfrak{h}$.

Now write $\mathfrak{g} = \mathbb{R}Y \oplus \mathfrak{h}$ and let

$v_0 \in V_X \setminus (0)$ be an eigenvector of $d\rho$.

Then it follows that

$$d\rho(Z)\, \mathbb{C}\cdot v_0 \subset \mathbb{C}\cdot v_0 \quad \forall Z \in \mathfrak{g}$$

and by connectedness of G we get

$$\rho(g)\, \mathbb{C} v_0 = \mathbb{C} v_0 \quad \forall g \in G$$

and hence G has a weight in V. $\square$