Now write $\mathfrak{g} = \mathbb{R}Y \oplus \mathfrak{h}$ and let $v_0 \in V_\chi \setminus (0)$ be an eigenvector of $d\rho(Y)$

Then it follows that

$$d\rho(Z)\, \mathbb{C}\cdot v_0 \subset \mathbb{C}\cdot v_0 \quad \forall Z \in \mathfrak{g}$$

and by connectedness of G we get

$$\rho(g)\, \mathbb{C} v_0 = \mathbb{C} v_0 \quad \forall g \in G$$

and hence G has a weight in V. $\square$

Proof of Thm 4.12

By recurrence on $\dim V$: by Thm 4.14 let $\chi : G \to \mathbb{C}^*$ be a weight of G in V and V_χ the corresponding weight space. Then $\dim(V/V_\chi) < \dim V$ and we obtain a representation of G in V/V_χ by setting : $\bar{\rho}(g)(v + V_\chi) = \rho(g)v + V_\chi$.

Let $e_1, \dots, e_f$ be a basis of V_χ and $e_{f+1}, \dots, e_n \in V$ such that $\bar{e}_\ell := e_\ell + V_\chi$, $f+1 \le \ell \le n$ is a basis of V/V_χ w.r.t. which $\bar{\rho}(g)$ is upper triangular $\forall g \in G$.

Then:
$f+1 \le i \le n$:
$$\bar{\rho}(g)\bar{e}_i = \chi_i(g)\bar{e}_i + \sum_{j=f+1}^{i-1} \bar{\rho}(g)_{ji}\bar{e}_j$$

and hence
$$\rho(g)e_i = \chi_i(g)e_i + \sum_{j=f+1}^{i-1} \bar{\rho}(g)_{ji}e_j \mod V_\chi$$

That is
$$\rho(g) = \begin{array}{cc} & f \\ f & \left(\begin{array}{ccc|cccc} \chi(g) & & & * & * & * \\ & \ddots & & * & * & * \\ & & \chi(g) & * & * & * \\ \hline & & & \chi_{f+1}(g) & & \bar{\rho}(g)_{ji} \\ & 0 & & & \ddots & \\ & & & & 0 & \ddots \\ & & & & & & \chi_n(g) \end{array}\right). \end{array}$$

Having seen that Lie groups that are solvable enjoy nice properties we turn to their characterization in terms of their Lie algebras.

Def. 4.16 A Lie algebra $\mathfrak{g}$ is solvable if there exists a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \triangleright \mathfrak{g}_1 \triangleright \cdots \triangleright \mathfrak{g}_r = (0)$$

with $\mathfrak{g}_i$ ideal in $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ abelian.

And the prototype of a solvable Lie algebra is: (see below)

Example 4.17

$$\mathfrak{g} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} : * \in \mathbb{R} \right\}.$$

Analogous to the case of groups we define for a Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{Linear Span of } \{[X,Y] : X, Y \in \mathfrak{g}\}$$

Def. 4.18 The derived series of a Lie algebra $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(\ell)} := (\mathfrak{g}^{(\ell-1)})^{(1)} = [\mathfrak{g}^{(\ell-1)}, \mathfrak{g}^{(\ell-1)}], \quad \ell \geqslant 2.$$

Remark 4.19 $\forall\, Z, X, Y \in \mathfrak{g}$: We have,

$$\mathrm{ad}(Z)([X,Y]) = [\mathrm{ad}(Z)X, Y] + [X, \mathrm{ad}(Z)Y]$$

Indeed (Jacobi) :

$$[\mathrm{ad}(Z)X, Y] + [X, \mathrm{ad}(Z)Y] = [[Z,X],Y] + [X,[Z,Y]]$$

$$= -[Y,[Z,X]] - [Y,[X,Z]] - [Z,[Y,X]]$$

$$= [Z,[X,Y]] = \mathrm{ad}(Z)([X,Y]).$$

As a result, $\mathfrak{g}^{(1)}$ is an ideal in $\mathfrak{g}$ and so are the $\mathfrak{g}^{(\ell)}$ $\forall \ell \geqslant 2$ by recurrence.

As in the case of groups (see Lemma 4.3) we have:

Lemma 4.20

(1) If $\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism we have:

$$\pi(\mathfrak{g}^{(i)}) = \pi(\mathfrak{g})^{(i)} \qquad \forall\, i \geq 1.$$

(2) Let $\mathfrak{r} \lhd \mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{r}$ is abelian $\Leftrightarrow \mathfrak{r} \supset \mathfrak{g}^{(1)}$.

<u>Proof</u>: (1) Follows from $\pi([X,Y]) = [\pi(X), \pi(Y)]$

(2) Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$ be the canonical projection homomorphism. Then by (1)

$$[\mathfrak{g}/\mathfrak{r}, \mathfrak{g}/\mathfrak{r}] = [\pi(\mathfrak{g}), \pi(\mathfrak{g})] = \pi(\mathfrak{g}^{(1)}) = 0$$

$\Leftrightarrow \mathfrak{r} \supset \mathfrak{g}^{(1)}$. □

As in the case of groups we have:

<u>Lemma 4.21</u> $\mathfrak{g}$ is solvable $\Leftrightarrow$ $\mathfrak{g}^{(r)} = (0)$ for some $r \geq 1$.

<u>Proof:</u>

($\Leftarrow$) We have

$$\mathfrak{g} = \mathfrak{g}_0 \triangleright \mathfrak{g}^{(1)} \triangleright \mathfrak{g}^{(2)} \cdots \triangleright \mathfrak{g}^{(r)} = (0)$$

and since $(\mathfrak{g}^{(i-1)})^{(1)} = \mathfrak{g}^{(i)}$ we have by lemma 4.20 (2) that $\mathfrak{g}^{(i-1)}/\mathfrak{g}^{(i)}$ is abelian.

($\Longrightarrow$) Let $\mathfrak{g} = \mathfrak{g}_0 \triangleright \mathfrak{g}_1 \triangleright \cdots \triangleright \mathfrak{g}_r = (0)$ with $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ abelian $1 \leq i \leq r$.

Since $\mathfrak{g}/\mathfrak{g}_1$ is abelian we have $\mathfrak{g}_1 \supset \mathfrak{g}^{(1)}$.

By recurrence, from $\mathfrak{g}_{i-1} \supset \mathfrak{g}^{(i-1)}$ we get since $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is abelian,

$$\mathfrak{g}_i \supset (\mathfrak{g}_{i-1})^{(1)} \supset (\mathfrak{g}^{(i-1)})^{(1)} = \mathfrak{g}^{(i)}. \quad \boxtimes$$

Def. 4.22 if $\mathfrak{g}$ is solvable we define

$$\mathrm{sol}(\mathfrak{g}) := \min\{ r \geq 1 : \mathfrak{g}^{(r)} = (0) \}.$$

Example 4.23 $\mathfrak{g} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} ; * \in \mathbb{R} \right\} \subset \mathfrak{gl}(n, \mathbb{R})$

then $\mathfrak{g}^{(1)} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \right\}$

$$\mathfrak{g}^{(2)} = \left\{ \begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix} \right\}$$

and $\mathfrak{g}^{(n)} = (0)$ so $\mathrm{sol}(\mathfrak{g}) = n$.

Here is a useful inductive criterion for solvability:

Lemma 4.24

(1) $\mathfrak{h} < \mathfrak{g}$, $\mathfrak{g}$ solvable $\Rightarrow$ $\mathfrak{h}$ solvable.

(2) $\mathfrak{h} \triangleleft \mathfrak{g}$, $\mathfrak{g}$ solvable $\Leftrightarrow$ $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Proof: (1) We have $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)}$ $\forall i \geq 1$ the statement then follows from lemma 4.21.

(2) Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ be the canonical projection. Then:

$$\pi(\mathfrak{g}^{(i)}) = (\mathfrak{g}/\mathfrak{h})^{(i)}, \quad \mathfrak{h}^{(k)} \subset \mathfrak{g}^{(k)}.$$

Thus if $\mathfrak{g}$ is solvable, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are solvable. Conversely let $m \geq 1$ with $0 = (\mathfrak{g}/\mathfrak{h})^{(m)} = \pi(\mathfrak{g}^{(m)})$. Then $\mathfrak{g}^{(m)} \subset \mathfrak{h}$ and if $\mathfrak{h}^{(r)} = 0$ we get $\mathfrak{g}^{(m+r)} = \{0\}$.

$\square$

Corollary 4.25 If $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{h}$ are solvable, $\operatorname{sol}(\mathfrak{g}) \leq \operatorname{sol}(\mathfrak{g}/\mathfrak{h}) + \operatorname{sol}(\mathfrak{h})$.

Now we come back to Lie groups and show:

Thm. 4.26 Let G be a connected Lie group. The following are equivalent:

(1) $\mathfrak{g} = \mathrm{Lie}(G)$ is solvable.

(2) G is a solvable group.

Proof:

(2) ⇒ (1) Let (Thm 4.7)

$$G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_r = (e)$$ with

G_i closed connected and G_{i-1}/G_i abelian $1 \leq i \leq r$. Let $\mathfrak{g}_i := \mathrm{Lie}(G_i) < \mathfrak{g}$.

Then by Cor. 3.70(2), $\mathfrak{g}_i \triangleleft \mathfrak{g}_{i-1}$, by Cor. 3.56, $\mathfrak{g}_{i-1}/\mathfrak{g}_i = \mathrm{Lie}(G_{i-1}/G_i)$ and since G_{i-1}/G_i is abelian, Prop. 3.55 implies $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ abelian.

(1) $\Rightarrow$ (2) By recurrence on $\mathrm{sol}(\mathfrak{g})$.

If $\mathrm{sol}(\mathfrak{g}) = 1$, $\mathfrak{g}$ is abelian and so is G since it is connected.

Let $r := \mathrm{sol}(\mathfrak{g}) \geq 2$. Then $(0) \neq \mathfrak{g}^{(r-1)} \triangleleft \mathfrak{g}$ and $\mathfrak{g}^{(r-1)}$ is abelian.

By Corollary 3.48(2), $\exp_G : \mathfrak{g}^{(r-1)} \longrightarrow G$ is a homomorphism and since $\mathfrak{g}^{(r-1)} \triangleleft \mathfrak{g}$ by the exercise series, $\exp_G(\mathfrak{g}^{(r-1)})$ is a normal subgroup of G. Thus ~~At $\exp_G(\mathfrak{g}$~~

$$N := \overline{\exp_G(\mathfrak{g}^{(r-1)})}$$

is normal, abelian and of course connected. Let $\mathfrak{n} = \mathrm{Lie}(N)$; by Cor. 3.70(2), $\mathfrak{n} \triangleleft \mathfrak{g}$, and $\mathfrak{n} \supset \mathfrak{g}^{(r-1)}$. Thus $\mathrm{solv}(\mathfrak{g}/\mathfrak{n}) \leq r-1$. Since $\mathfrak{g}/\mathfrak{n} = \mathrm{Lie}(G/N)$ we are done by recurrence with the same argument than in the proof of Thm 4.7. ∎

~~by the same argument than in the proof of Thm 4.7. ∎~~

This justifies the following definition:

Def. 4.27 A connected solvable Lie group is $\underbrace{\text{a Lie}}_{\text{connected}}$ group G with $\mathfrak{g} = \mathrm{Lie}(G)$ solvable.

The same strategy of proof than in Lie's theorem (Thm 4.12) works to show:

Thm 4.28 Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of a solvable Lie algebra $\mathfrak{g}$ into a complex vector space V. Then there is a basis of V wrt which all $\rho(X)$, $X \in \mathfrak{g}$, are upper triangular.

4.2. Nilpotent Lie algebras and Lie groups.

Nilpotent Lie algebras play a pivotal role in the structure theory of Lie algebras. On one hand they lead to powerful characterizations of solvable algebras, on the other hand one of those characterizations (Cartan's criterion) plays a key role in the characterization of semisimple algebras which are situated at the other end of the structural spectrum.

In this section we will place the emphasis on nilpotent Lie algebras and shortly indicate the relation with nilpotent Lie groups.

Let $\mathfrak{g}$ be a Lie algebra. We will need the following generalisation of $[\mathfrak{g},\mathfrak{g}]$:

Let $\mathfrak{a}, \mathfrak{b}$ be subspace of $\mathfrak{g}$.

$[\mathfrak{a}, \mathfrak{b}] := \text{Linear span}\{[X, Y] : X, Y \in \mathfrak{g}\}$.

We observe

Lemma 4.29:

(1) If $\mathfrak{a} \triangleleft \mathfrak{g}$ and $\mathfrak{b} \triangleleft \mathfrak{g}$ then $[\mathfrak{a}, \mathfrak{b}] \triangleleft \mathfrak{g}$.

(2) Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a Lie algebra homomorphism and $\mathfrak{a}, \mathfrak{b}$ subspaces of $\mathfrak{g}$. Then $\pi([\mathfrak{a}, \mathfrak{b}]) = [\pi(\mathfrak{a}), \pi(\mathfrak{b})]$.

Proof (2) Clear.

(1) Follows from the identity:

$$\mathrm{ad}(X)([Y, Z]) = [\mathrm{ad}(X)Y, Z] + [Y, \mathrm{ad}(X)Z]$$

$\forall X, Y, Z \in \mathfrak{g}$.

$\square$

Def. 4.30 : $\mathfrak{g}$ is nilpotent if there is a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_r = (0)$$

of subspaces such that $[\mathfrak{g}, \mathfrak{g}_{i-1}] \subset \mathfrak{g}_i$

$1 \leq i \leq r$.

Clearly such $\mathfrak{g}_i$'s are then ideals of $\mathfrak{g}$.

As for the meaning of this condition observe that if $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_i$ is the canonical projection we have

$$\left[\mathfrak{g}/\mathfrak{g}_i, \mathfrak{g}_{i-1}/\mathfrak{g}_i\right] = \pi([\mathfrak{g}, \mathfrak{g}_{i-1}]) \subset \pi(\mathfrak{g}_i) = (0)$$

that is

$$\mathfrak{g}_{i-1}/\mathfrak{g}_i \subset Z(\mathfrak{g}/\mathfrak{g}_i).$$

Now as in the case of solvable algebras, define for a Lie algebra $\mathfrak{g}$:

$$C^1(\mathfrak{g}) = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$$

and inductively $\quad C^i(\mathfrak{g}) = [\mathfrak{g}, C^{i-1}(\mathfrak{g})], \; i \geq 2.$

Def. 4.31 $C^{\ell}(\mathfrak{g})$, $\ell \geq 1$ is called the central series of $\mathfrak{g}$.

It has the property that

$$(*) \qquad C^{i-1}(\mathfrak{g})/C^{i}(\mathfrak{g}) \subset Z\left(\mathfrak{g}/C^{i}(\mathfrak{g})\right).$$

Prop. 4.31: The following are equivalent:

(1) $\mathfrak{g}$ is nilpotent.

(2) $C^{r}(\mathfrak{g}) = 0$ for some $r \geq 1$.

(3) $\exists\, m \geq 1$ s.t.
$\mathrm{ad}(x_1) \cdots \mathrm{ad}(x_p) = 0 \quad \forall x_1, \dots, x_p \in \mathfrak{g}$.

Proof: (1) $\Rightarrow$ (2): $\mathfrak{g}/\mathfrak{g}_1$ abelian $\Rightarrow$ $\mathfrak{g}_1 \supset C^{1}(\mathfrak{g})$.

By recurrence: $\mathfrak{g}_{i-1} \supset C^{i-1}(\mathfrak{g})$ hence

$$\mathfrak{g}_i \supset [\mathfrak{g}, \mathfrak{g}_{i-1}] \supset [\mathfrak{g}, C^{i-1}(\mathfrak{g})] = C^{i}(\mathfrak{g}).$$

(2) $\Rightarrow$ (1): clear from the definition.

(2) $\Leftrightarrow$ (3) clear since $C^k(\mathfrak{g})$ is the linear span of $\mathrm{ad}(X_1)\circ\cdots\circ\mathrm{ad}(X_k)(Y)$ $X_1, \dots, X_k, Y \in \mathfrak{g}$. $\boxed{}$

Def. 4.31½: If $\mathfrak{g}$ is nilp., $\mathrm{nil}(\mathfrak{g}) = \min\{r \geq 1 : C^r(\mathfrak{g}) = (0)\}$.

Examples 4.32 (1) A nilpotent Lie algebra is solvable. This follows from $C^k(\mathfrak{g}) \supset \mathfrak{g}^{(k)}$ $k \geq 1$.

(2) $$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} : * \in \mathbb{R} \right\} \subset \mathfrak{gl}(n, \mathbb{R}).$$

Then $$C^1(\mathfrak{n}) = \left\{ \begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & & & 0 \end{pmatrix} \right\}$$

and $C^n(\mathfrak{n}) = \{(0)\}$.

Next we show a slightly surprising fact which follows from Lie's theorem.

Thm 4.33: $\mathfrak{g}$ is solvable $\Leftrightarrow$ $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Remark : Let $\mathfrak{g}$ be nilpotent with $nil(\mathfrak{g}) = r$.

Then $C^r(\mathfrak{g}) = (0)$ and (*)

$$0 \neq C^{r-1}(\mathfrak{g}) \subset Z(\mathfrak{g}).$$

In particular if $\mathfrak{g}$ is nilpotent with $\mathfrak{g} \neq (0)$ then $Z(\mathfrak{g}) \neq (0)$.

We will need:

Lemma 4.34 $\mathfrak{h} \triangleleft \mathfrak{g}$.

(1) $\mathfrak{g}$ nilpotent $\Rightarrow$ $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ nilp.

(2) If $\mathfrak{g}/\mathfrak{h}$ is nilpotent and $\mathfrak{h} \subset \mathfrak{z}(\mathfrak{g})$ $\mathfrak{g}$ is nilpotent.

Proof: (1) follows from $C^j(\mathfrak{h}) \subset C^j(\mathfrak{g})$ and $\pi(C^j(\mathfrak{g})) = C^j(\mathfrak{g}/\mathfrak{h})$ where $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection.

(2) Let $r \geq 1$ with

$$\pi(C^r(\mathfrak{g})) = C^r(\mathfrak{g}/\mathfrak{h}) = (0)$$

So $C^r(\mathfrak{g}) \subset \mathfrak{h} \subset \mathfrak{z}(\mathfrak{g})$ and hence $C^{r+1}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{h}] = (0)$. $\blacksquare$

Proof of Thm 4.33

($\Leftarrow$) $\mathfrak{g}^{(r+1)} = (\mathfrak{g}^{(1)})^{(r)} \subset C^r(\mathfrak{g}^{(1)})$

so that if $\mathfrak{g}^{(1)}$ is nilpotent, $\mathfrak{g}$ is solvable

Remark: $\mathfrak{g} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : * \in \mathbb{R} \right\}$

$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} : * \in \mathbb{R} \right\}$

then $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are abelian, in particular nilpotent but $\mathfrak{g}$ is not nilpotent since $Z(\mathfrak{g}) = (0)$.

$(\Longrightarrow)$ Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

We consider

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] & \mathfrak{gl}(\mathfrak{g}) \arrow[r] & \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}) \\
X \arrow[r, mapsto] \arrow[rr, bend right, "\mathrm{ad}_{\mathbb{C}}(X)."'] & \mathrm{ad}(X) \arrow[r, mapsto] & \mathrm{ad}(X) \otimes 1.
\end{tikzcd}$$

By Thm 4.28 let $e_1, \dots, e_m$ be a basis of $\mathfrak{g}_{\mathbb{C}}$ such that:

$$\mathrm{ad}_{\mathbb{C}}(\mathfrak{g}) \subset \left\{ \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & * \end{pmatrix} : * \in \mathbb{C} \right\}.$$

Then $\mathrm{ad}_{\mathbb{C}}([\mathfrak{g}, \mathfrak{g}]) \subset \left\{ \begin{pmatrix} 0 & \cdots & * \\ 0 & \ddots & 0 \end{pmatrix} : * \in \mathbb{C} \right\}$.

Hence $\mathrm{ad}_{\mathbb{C}}([\mathfrak{g}, \mathfrak{g}])$ is nilpotent. But

$$\mathfrak{n} = \mathrm{Ker}\, \mathrm{ad}_{\mathbb{C}}\big|_{[\mathfrak{g},\mathfrak{g}]} \subset Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$$
$$\subset Z([\mathfrak{g}, \mathfrak{g}])$$

Which by lemma 4.34(2) implies $[\mathfrak{g}, \mathfrak{g}]$ nilpotent. $\square$