Solution 1

1. Let G be a Hausdorff topological group, and $H < G$ a subgroup.
   (a) Show that if H is closed and has finite index in G, then H is open.
   (b) Show that if H is abelian, then so is its closure $\overline{H}$.
   (c) Recall that a group G is *solvable* if there exists a chain $G \triangleright G_1 \triangleright G_2 \dots G_n \triangleright (e) = G_{n+1}$ with $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} abelian for all $0 \leqslant i \leqslant n$. Show that if G is solvable then one can also find a chain of subgroups G_i as above with G_i closed in G.

2. (a) Find an injection $O(1,1) \hookrightarrow O(p,q)$ for $p,q \geqslant 1$.
   (b) Show that the topological group $O(p,q)$ for $p,q \geqslant 1$ is not compact.
   (c) Show that $O(1,1)$ has four connected components.

   Hints:

   (a) Use the definition of $O(p,q)$ and block matrices.
   (b) Give an explicit unbounded sequence in $O(1,1)$.
   (c) Observe that $\mathrm{SO}(1,1)$ has index 2 in $O(1,1)$ and 2 connected components using an explicit parametrization and its action on $\mathbb{R}^2$.

3. (a) Let X be a *compact* Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.
   (b) The objective of this exercise is to show that $(\mathrm{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.
   Consider the "middle thirds" Cantor set
   $$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$
   in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1 - 3^{-n}, 1]$. Further we construct a sequence of homeomorphisms $h_n \in \mathrm{Homeo}(C)$ as follows:
   - $h_n(x) = x$ for all $x \in C \setminus (U_n \cup V_n)$,
   - $h_n(0) = 0$,
   - $h_n(U_{n+1}) = U_n$,
   - $h_n(U_n \setminus U_{n+1}) = V_{n+1}$,
   - $h_n(V_n) = V_n \setminus V_{n+1}$.

These restrict to homeomorphisms $h_n|_X$ on $X := C \setminus \{0\}$.

Show that the sequence $(h_n|_X)_{n\in\mathbb{N}} \subset \mathrm{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n\in\mathbb{N}} \subset \mathrm{Homeo}(X)$ of their inverses does not!

Remark: However, if X is locally compact and *locally connected* then $\mathrm{Homeo}(X)$ is a topological group.

Solution:

(a) Denote by $m : \mathrm{Homeo}(X) \times \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ the composition $m(f,g) = f \circ g$ and by $i : \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ the inversion $i(f) = f^{-1}$. We need to see that m and i are continuous.

i. m is continuous: We want to show that m is continuous at any tuple $(f,g) \in \mathrm{Homeo}(X) \times \mathrm{Homeo}(X)$. Thus let $S(K,U) \ni f \circ g$ be a subbasis neighborhood of $f \circ g$, i.e. $K \subset X$ is compact and $U \subset X$ is open such that $f(g(K)) \subset U$. Observe that $g(K)$ is compact and is contained in $f^{-1}(U)$ which is open. Because X is (locally) compact we may find an open set $V \subset X$ with compact closure $\overline{V}$ such that

$$g(K) \subset V \subset \overline{V} \subset f^{-1}(U).$$

It is now easy to verify that $W := S(\overline{V}, U) \times S(K, V)$ is an open neighborhood of (f,g) such that $m(W) \subset S(K,U)$. Indeed, (f,g) is by construction of V contained in W and for any $(h_1, h_2) \in W$ we get

$$h_2(K) \subset V \subset \overline{V} \subset h_1^{-1}(U).$$

Hence, m is continuous at every point of $\mathrm{Homeo}(X) \times \mathrm{Homeo}(X)$.

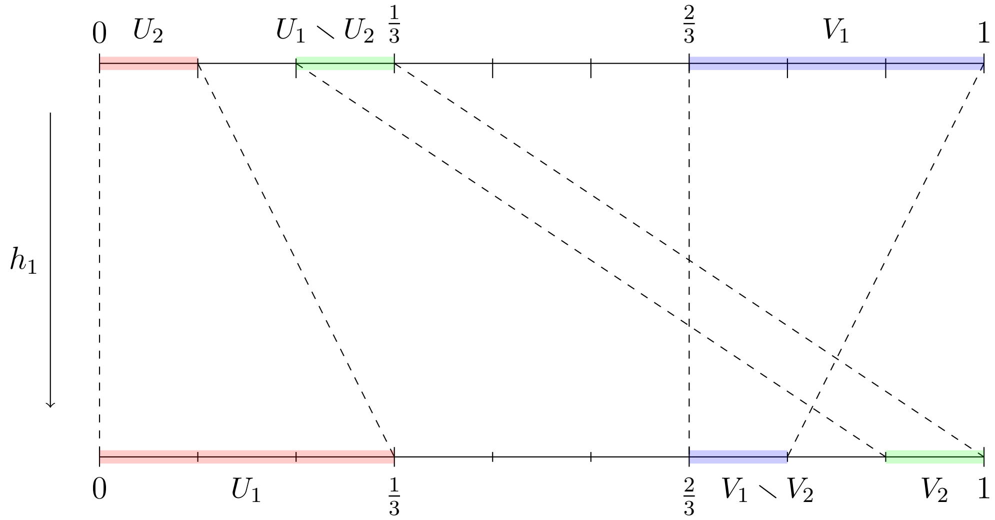
ii. i is continuous: Let $f \in \mathrm{Homeo}(X)$, $K \subset X$ compact and $U \subset X$ open. Then

$$\begin{aligned} i(f) \in S(K,U) &\iff f^{-1}(K) \subset U \iff K \subset f(U) \\ &\iff f(U^c) = f(U)^c \subset K^c \iff f \in S(U^c, K^c). \end{aligned}$$

Observe that U^c is compact as a closed subset of the compact space X and that K^c is open as the complement of a (compact) closed set.

This shows that $i^{-1}(S(K,U)) = S(U^c, K^c)$ for every element $S(K,U)$ of a subbasis for the compact-open topology on $\mathrm{Homeo}(X)$, whence i is continuous.

(b) The following picture gives a pictorial description of what h_1 does on the Cantor set C.



Since $h_n(0) = 0$ we obtain indeed a homeomorphism $h_n|_X \in \mathrm{Homeo}(X)$ by restriction to $X = C \setminus \{0\}$. Let us first see that the sequence $(h_n|_X)_{n\in\mathbb{N}}$ indeed converges to $\mathrm{id} \in \mathrm{Homeo}(X)$. For that let $S(K, U)$ be a subbasis neighborhood of id, i.e. K is a compact subset of X contained in some open set $U \subset X$. Therefore we can find an $M \in \mathbb{N}$ such that U_M and K are disjoint.

If $1 \notin K$ then there is also an $N \geqslant M$ such that V_n and K are disjoint. In this case $h_n|K$ is the identity and hence in $S(K, U)$ for all $n \geqslant N$.

If $1 \in K$ then there is an $N \geqslant M$ such that V_N is contained in U. Consequently, we have

$$h_n(K \setminus V_n) = K \setminus V_n, \quad h_n(K \cap V_n) \subset V_n \subset V_N \subset U,$$

for all $n \geqslant N$.

In any case the sequence $(h_n|_X)_{n\in\mathbb{N}}$ will be in $S(K, U)$ for large enough n such that $\lim_{n\to\infty} h_n|_X = \mathrm{id}$. On the other hand $h_n^{-1}(1) \in U_n$ for every $n \in \mathbb{N}$ such that $\lim_{n\to\infty} h_n^{-1}(1) = 0$. Thus the sequence $(h_n^{-1}|_X)_{n\in\mathbb{N}}$ certainly does not converge to id.

Remark: Note that we actually needed to remove 0 from C for this construction to work. In fact, the sequence h_n does not converge to id in $\mathrm{Homeo}(C)$: Let $K = [0, 1/9] \cap C, U = [0, 1/2) \cap C$. Then $S(K, U)$ is again a neighborhood of id. However, $U_n \subset K$ for every $n \geqslant 2$ and $V_{n+1} \subset U^c$ which implies that

$$h_n(U_n \setminus U_{n+1}) \subset U^c,$$

i.e. $h_n \notin S(K, U)$.

4. Show that if M is a manifold of dimension at least one, then $\mathrm{Homeo}(M)$ is not locally compact.

Solution: We will prove that Homeo(M) is not locally compact for any compact manifold M. Note that we can think of M as a compact metric space (M, d) by Urysohn's metrization theorem. In the case when M is a smooth manifold this is even easier to see by endowing it with a Riemannian metric. This puts us now in the favorable position of being able to identify the compact-open topology on Homeo(X) with the topology of uniform convergence.

We denote by

$$d_\infty(f,g) := \sup\{d(f(x), g(x)) : x \in M\}$$

the metric of uniform convergence on Homeo(M). Further denote by $B_f^\infty(r)$ the ball of radius $r > 0$ about a homeomorphism $f \in$ Homeo(M). In order to show that Homeo(M) is not locally compact we will construct in every $\varepsilon > 0$ ball about the identity $B_{\mathrm{id}}^\infty(\varepsilon)$ a sequence of homeomorphisms $(f_k)_{k\in\mathbb{N}}$ with no convergent subsequence.

Let $\varepsilon > 0$ and denote $B = B_{\mathrm{id}}^\infty(\varepsilon)$. Further, let $x_0 \in M$ and choose a coordinate chart $\varphi : U \subset B_{\varepsilon/2}(x_0) \to \mathbb{R}^n$ centered at x_0 (i.e. $\varphi(x_0) = 0$) contained in the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ about x_0 in M. Consider the homeomorphisms

$$\psi_k : \overline{B_1}(0) \to \overline{B_1}(0), x \mapsto \|x\|^k x$$

on the closed unit ball $\overline{B_1}(0)$ in $\mathbb{R}^n$ which fix $0 \in \mathbb{R}^n$ and the boundary n-sphere pointwise. Note that the sequence $(\psi_k)_{k\in\mathbb{N}}$ converges pointwise to

$$\psi_\infty = \begin{cases} x, & \text{if } x \in \partial B_1(0), \\ 0, & \text{if } x \in B_1(0). \end{cases}$$

Now, define

$$f_k(x) := \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ \varphi^{-1}(\psi_k(\varphi(x))), & \text{if } x \in \varphi^{-1}(B_1(0)). \end{cases}$$

It is easy to see that the maps $f_k : M \to M$ are indeed homeomorphisms: $f_k|_{\varphi^{-1}(\overline{B_1}(0))^c} = \mathrm{id} : \varphi^{-1}(\overline{B_1}(0))^c \to \varphi^{-1}(\overline{B_1}(0))^c$ is a homeomorphism, $\varphi^{-1} \circ \psi_k \circ \varphi : \varphi^{-1}(\overline{B_1}(0)) \to \varphi^{-1}(\overline{B_1}(0))$ is a homeomorphism and both coincide on $\varphi^{-1}(\partial B_1(0))$.

Further, the homeomorphisms f_k map the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ to itself and fix x_0. Therefore,

$$d(f_k(x), x) \leqslant d(f_k(x), \underbrace{f_k(x_0)}_{=x_0}) + d(x_0, x) < \varepsilon,$$

for every $x \in B_{\varepsilon/2}(x_0)$, and clearly $f_k(x) = x$ for every $x \notin B_{\varepsilon/2}(x_0)$. Hence, the sequence $(f_k)_{k\in\mathbb{N}}$ is in $B_\varepsilon^\infty(\mathrm{id})$.

However, the sequence $(f_k)_{k\in\mathbb{N}}$ converges pointwise to

$$f_\infty(x) = \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ x_0, & \text{if } x \in \varphi^{-1}(B_1(0)), \end{cases}$$

If there were a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ converging to some $f \in \mathrm{Homeo}(M)$ uniformly then this sequence would also converge pointwise to f, i.e. f needs to coincide with f_∞. But f_∞ is not even continuous which contradicts our assumption of $f \in \mathrm{Homeo}(M)$. Therefore $(f_k)_{k\in\mathbb{N}} \subset B_\varepsilon^\infty(\mathrm{id})$ has no uniformly convergent subsequences.

5. Let (X, d) be a proper metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X\}.$$

   (a) Show that Iso(X) $\subset \mathrm{Homeo}(X)$ is locally compact with respect to the compact-open topology.

   (b) Show that if additionally X is compact, then $\mathrm{Iso}(X)$ is compact.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem.

Solution:

   (a)

   (b) The compact-open topology on $\mathrm{Homeo}(X)$ coincides with the topology induced by the metric of uniform convergence

$$d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

Note that by Arzelà–Ascoli a family $\mathcal{F} \subseteq C(X, X)$ of continuous maps is compact if and only if $\mathcal{F}$ is equicontinuous, and $\mathcal{F}$ is closed. Equicontinuity of $\mathcal{F} := \mathrm{Iso}(X)$ is clear, because we are dealing with isometries. We check that $\mathrm{Iso}(X)$ is closed. Let $f \in C(X, X)$ and let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Iso}(X)$ be a sequence converging to it. Let $x, y \in X$ then

$$\begin{aligned}
0 &\leqslant |d(f(x), f(y)) - d(x, y)| \\
&= |d(f(x), f(y)) - d(f_n(x), f_n(y))| \\
&\leqslant |d(f(x), f(y)) - d(f_n(x), f(y))| + |d(f_n(x), f(y)) - d(f_n(x), f_n(y))| \\
&\leqslant d(f(x), f_n(x)) + d(f(y), f_n(y)) \to 0 \quad (n \to \infty).
\end{aligned}$$

Hence, f is an isometry as wished for. Because f was arbitrary this shows that $\mathrm{Iso}(X) \subseteq C(X, X)$ is closed.