Solution 2

1. Let G be a topological group.

   (a) Show that if $U \ni e$ is a neighborhood of the neutral element e, then there exists $V \ni e$ symmetric and open with $V^2 \subset U$.

   (b) Use (a) to show that if e is closed, then G is Hausdorff.

   Solution:

   (a) Take $W := W_1 \cap W_2$, with W_1, W_2 open in G and $W_1 \times W_2 \subset m^{-1}(e, e)$. Then symmetrize as in class.

   (b) For $g \neq h \in G$ take an open neighbourhood U of e that does not contain $g^{-1}h$, and V as in (a). Then gV and hV separate g and h.

2. Let H be a topological group and $p\colon G \to H$ a covering map, where à priori G is just a topological space. Show that then for every choice $e_G \in p^{-1}(e_H)$ there exists a unique topological group structure on G with neutral element e_G such that $p\colon G \to H$ is a homomorphism.

   Solution:

   We will lift the multiplication and inversion maps to G and show that they define a group structure on G.

   Let $m : H \times H \to H$ and $i : H \to H$ denote the multiplication and inversion maps of H, respectively, and let e_G be an arbitrary element of the fiber $p^{-1}(e_H) \subseteq G$. Since $p \times p : G \times G \to H \times H$ is a covering satisfying $(m \circ (p \times p))_*(\pi_1(G, e_G)) \subseteq \pi_1(H, e_H)$ the map $m \circ (p \times p) : G \times G \to H$ has a unique lift $\tilde{m} : G \times G \to G$ satisfying $\tilde{m}(e_G, e_G) = e_G$ and $p \circ \tilde{m} = m \circ (p \times p)$:

$$\begin{array}{ccc} G \times G & \xrightarrow{\tilde{m}} & G \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ H \times H & \xrightarrow{m} & H \end{array}$$

   Because p is a local isomorphism and $p \circ \tilde{m} = m \circ (p \times p)$ is continuous also $\tilde{m}$ is continuous. By the same reasoning, $i \circ p : G \to H$ has a smooth lift $\tilde{i} : G \to G$ satisfying $\tilde{i}(e_G) = e_G$ and $p \circ \tilde{i} = i \circ p$:

$$\begin{array}{ccc} G & \xrightarrow{\tilde{i}} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ H & \xrightarrow{i} & H \end{array}$$

We define multiplication and inversion in G by $xy = \tilde{m}(x, y)$ and $x^{-1} = \tilde{i}(x)$. By the above commutative diagrams we obtain

$$p(xy) = p(x)p(y), \qquad p(x^{-1}) = p(x)^{-1}.$$

It remains to show that G is a group with these operations, for then it is a topological group because $\tilde{m}$ and $\tilde{i}$ are continuous and the above relations imply that p is a homomorphism.

First we show that e_G is an identity for multiplication in G. Consider the map $f : G \to G$ defined by $f(x) = e_G x$. Then

$$p(f(x)) = p(e_G)p(x) = e_H p(x) = p(x),$$

so f is a lift of $p : G \to H$. The identity map id_G is another lift of p, and it agrees with f at a point because $f(e_G) = \tilde{m}(e_G, e_G) = e_G$, so the unique lifting property of covering maps implies that $f = \mathrm{id}_G$, or equivalently, $e_G x = x$ for all $x \in G$. The same argument shows that $xe_G = x$.

Next, to show that multiplication in G is associative, consider the two maps $\alpha_L, \alpha_R : G \times G \times G \to G$ defined by

$$\alpha_L(x, y, z) = (xy)z, \quad \alpha_R(x, y, z) = x(yz).$$

Then

$$p(\alpha_L(x, y, z)) = (p(x)p(y))p(z) = p(x)(p(y)p(z)) = p(\alpha_R(x, y, z)),$$

so α_L and α_R are both lifts of the same map $\alpha(x, y, z) = p(x)p(y)p(z)$. Because α_L and α_R agree at (e_G, e_G, e_G), they are equal. A similar argument shows that $x^{-1}x = xx^{-1} = e_G$, so G is a group.

The uniqueness follows from the uniqueness of the lifting property.

3. Show that if $p\colon G \to H$ is a covering homomorphism of topological groups, then there is a local isomorphism from G to H and from H to G.

   Solution: Because p is a continuous covering map there are open neighbourhoods $U \subseteq H$ of e_H and $V \subseteq G$ of e_G such that $p|_V : V \to U$ is an isomorphism. Thus $(p|_V, V)$ is a local isomorphism from G to H. Conversely,

4. Show that the local homomorphism in the example at the beginning of Chapter 2.4 cannot be extended to a global continuous homomorphism.

   Solution: It would have to be given by the same formula, but one easily sees that it is not continuous at -1.

5. Let G be a locally compact Hausdorff topological group and $\Lambda : C_{oo}(G) \to \mathbb{C}$ a positive linear functional that is represented by the regular Borel measure μ. Let $g \in G$. Show that the positive linear function $\lambda(g)^*\Lambda$ is represented by $g_*\mu$.

   Solution: Computation.

6. Define what a right (invariant) Haar functional on a locally compact group is. Prove that it always exists and that it is unique up to scalar multiple in $\mathbb{R}_{>0}$. Refer to page -2-48- in the notes.

   Solution: If Λ is a left Haar functional, then $\Lambda'(f) := \Lambda(\check{f})$ is a right Haar functional, where $\check{f}(g) := f(g^{-1})$. Note that $(\widecheck{\rho(g^{-1})f}) = \lambda(g)\check{f}$, since

$$(\widecheck{\rho(g^{-1})f})(h) = (\rho(g^{-1})f)(h^{-1}) = f(h^{-1}g) = \check{f}(g^{-1}h) = (\lambda(g)\check{f})(h).$$

   Thus

$$(\rho^*(g)\Lambda')(f) = \Lambda'(\rho(g^{-1})f) = \Lambda(\widecheck{\rho(g^{-1})f})) = \Lambda(\lambda(g)\check{f}) = \Lambda(\check{f}) = \Lambda'(f).$$