Solution 3

1. Let $H < G$ be a subgroup of a topological group. Show that the action $G \times G/H \to G/H$ is continuous.

   Solution: Since multiplication and projection are continuous, the map

   $$\varphi : G \times G \to G/H,\ (g, g') \mapsto gg'H$$

   is continuous. Furthermore, the map

   $$\psi : G \times G \to G \times G/H,\ (g, g') \mapsto (g, g'H)$$

   is open, since both the projection and the identity are open. Denote by a the action map $a : G \times G/H \to G/H$. If $U \subseteq G/H$ is open, then $a^{-1}(U) = \psi(\varphi^{-1}(U))$ is open.

2. In the setting of Example 2.44 (1), show that if $1 \leqslant k \leqslant n-1$ then $\mathrm{SO}(n, \mathbb{R})$ acts transitively on GO_k.

   Solution: Extend orthonormal vectors $v_1, \dots, v_k \in GO_k$ to an orthonormal basis $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ of $\mathbb{R}^n$. If $\det(v_1 | \dots | v_n) = -1$, replace v_n by $-v_n$. Then $M := \left(v_1 \,\middle|\, \dots \,\middle|\, v_n \right)$ is in $\mathrm{SO}(n, \mathbb{R})$ and hence $M^{-1}(v_1, \dots, v_k) = (e_1, \dots, e_k)$.

3. In the setting of Example 2.44 (2), let $P := P_{(1,\dots,n-1)}$ be the subgroup of $\mathrm{GL}(n, \mathbb{R})$ of upper triangular matrices. Show that $\mathrm{GL}(n, \mathbb{R})/P$ is compact and deduce that $\mathrm{GL}(n, \mathbb{R})/P_d$ is compact as well.

   Solution: We show that $O(n, \mathbb{R})$ acts transitively on $\mathrm{Gr} := \mathrm{GL}(n, \mathbb{R})/P$. If $V = V_1 \subseteq \dots \subseteq V_{n-1}$ is a complete flag in Gr, we can choose $0 \neq v_1 \in V_1$ of norm one. Then, choose $0 \neq v_2 \in V_2$ of norm one such that $V_2 = \langle v_1, v_2 \rangle$ and v_1 and v_2 are orthogonal. Continue to find $(v_1, \dots, v_n)$ orthonormal and such that $V_i = \langle v_1, \dots, v_i \rangle$ for all $i = 1, \dots, n$. Then $M := \left(v_1 \,\middle|\, \dots \,\middle|\, v_n \right)$ is in $O(n, \mathbb{R})$ and $M^{-1}V = E$, where E is the standard flag. Thus the orbit map $O(n, \mathbb{R}) \to \mathrm{Gr}$, $M \mapsto ME$ is continuous and surjective, and since $O(n, \mathbb{R})$ is compact, so is Gr.

   Every partial flag can be extended to a full flag, and the same argument shows that $O(n, \mathbb{R})$ acts transitively on Gr_d.

4. $\mathrm{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ and the orbit map

   $$\mathrm{SL}(2, \mathbb{R})/N \to \mathbb{R}^2 \setminus \{0\}, \quad gN \mapsto g(e_1),$$

where $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$ is an $\mathrm{SL}(2,\mathbb{R})$-equivariant homeomorphism. Using this show that there is an $\mathrm{SL}(2,\mathbb{R})$-invariant regular Borel measure on $\mathrm{SL}(2,\mathbb{R})/N$.

Solution: On $\mathbb{R}^2 \smallsetminus \{0\}$ the restriction of the Lebesgue measure on $\mathbb{R}^2$ is an $\mathrm{SL}(2,\mathbb{R})$-invariant regular Borel measure λ. Thus we can define a measure on $\mathrm{SL}(2,\mathbb{R})/N)$ by setting $\mu := h^*\lambda$, where h is the above homeomorphism. Because the homeomorphism is equivariant, μ will be invariant, since λ is.

5. $\mathrm{SL}(2,\mathbb{R})$ acts transitively on the projective line $\mathbb{P}^1(\mathbb{R})$ and let

$$B = \mathrm{Stab}(\mathbb{R}e_1) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

Show that on $\mathrm{SL}(2,\mathbb{R})/B$ there is no $\mathrm{SL}(2,\mathbb{R})$-invariant regular Borel measure.

Solution: Recall that $G = \mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\} \subset \hat{\mathbb{C}}$ and its boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}}$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

Note that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\partial\mathbb{H}$ and the stabilizer of ∞ is the subgroup of upper triangular matrices P. We may therefore identify $G/P \cong \mathbb{R} \cup \{\infty\}$.

Suppose there is a finite G-invariant measure m on $G/P \cong \mathbb{R} \cup \{\infty\}$. Consider the restriction $\mu = m|_{\mathbb{R}}$ of this measure to the real line. Observe that G acts on the real line via translations

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} .\xi = \xi + t, \qquad \xi, t \in \mathbb{R},$$

such that μ is in particular a translation invariant measure on $\mathbb{R}$, i.e. μ is a Haar measure on $\mathbb{R}$. By uniqueness of Haar measures μ must be a multiple of the Lebesgue measure on $\mathbb{R}$. Since m is finite μ is the zero measure. That means that m is a positive multiple of the dirac measure at ∞, i.e. $m = \lambda \cdot \delta_\infty$ for some $\lambda > 0$. Now consider the rotation

$$i(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .z = -\frac{1}{z}$$

that sends ∞ to 0. By G-invariance we must have

$$\lambda \cdot \delta_\infty = i_*(\lambda \cdot \delta_\infty) = \lambda \cdot \delta_0$$

such that $\lambda = 0$; in contradiction to our assumption.