Solution 4

1. Let T_ℓ, $\ell \geqslant 3$, be the ℓ-regular tree together with the combinatorial distance d. Show that $\mathrm{Iso}(T_\ell, d)$ is not a Lie group.

   Hint: Try to show that the stabilizer of a vertex is a profinite group in an explicit way, and thus totally disconnected. Deduce that $\mathrm{Iso}(T_\ell, d)$ is not a manifold.

   You could also show that every neighborhood of the identity in $\mathrm{Iso}(T_\ell, d)$ contains a non-trivial subgroup, which contradicts exercise 6 on exercise sheet 6.

2. Let G be a Lie group, $H < G$ a subgroup that is also a regular submanifold.

   (a) Show that H is a Lie group.

   (b) Prove that H is a closed subgroup of G.

   Solution:

   (b) Let $x \in \overline{H}$. As G is clearly first countable, we find $(x_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $x = \lim_{n\to\infty} x_n$. Let $V \subseteq W \subseteq \overline{W} \subseteq U$ open neighbourhoods of $1 \in G$ with compact closure and assume that $\psi : U \to (-1,1)^{\dim G}$ is a chart as in the definition of a regular submanifold. Assume furthermore that V is symmetric and $VV \subseteq W$. By assumption, there is $N \geqslant 1$ such that $x_n \in xV$ for all $n \geqslant N$, thus $x_N^{-1}x_n \in V^{-1}x^{-1}xV = VV \subseteq W$ for all $n \geqslant N$, and thus $x_N^{-1}x_n \in H \cap VV \subseteq H \cap \overline{W}$. We note that $H \cap \overline{W}$ is compact by the choice of U. Indeed, $\psi(\overline{W}) \subsetneq (-1,1)^{\dim G}$ is compact, and so is $\psi(\overline{W}) \cap \{0\}^{\dim G - \dim H} \times (-1,1)^{\dim H}$. But $x_N^{-1}x_n$ is convergent and has a limit y in $H \cap \overline{W}$; whence $x_N y = x \in H$.

3. We consider the determinant function $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \mathrm{tr}.$$

Solution: Let $A \in \mathbb{R}^{n\times n} \cong T_I \operatorname{GL}(n,\mathbb{R})$. We compute

$$
\begin{aligned}
D_I \det(A) &= \frac{d}{dt}\bigg|_{t=0} \begin{vmatrix} 1+ta_{1,1} & ta_{1,2} & \cdots & ta_{1,n} \\ ta_{2,1} & 1+ta_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,1} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \\
&\overset{(*)}{=} \frac{d}{dt}\bigg|_{t=0} \left((1+ta_{1,1}) \begin{vmatrix} 1+ta_{2,2} & ta_{2,3} & \cdots & ta_{2,n} \\ ta_{3,2} & 1+ta_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,2} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \right) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+1} \frac{d}{dt}\bigg|_{t=0} \left(ta_{2,j} \begin{vmatrix} ta_{1,2} & \cdots & ta_{1,n} \\ & * & \end{vmatrix} \right) \\
&\overset{(**)}{=} \left(a_{1,1} \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} + \frac{d}{dt}\bigg|_{t=0} \begin{vmatrix} 1+ta_{2,2} & ta_{2,3} & \cdots & ta_{2,n} \\ ta_{3,2} & 1+ta_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,2} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \right) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+1} \left(a_{2,j} \begin{vmatrix} 0\cdot a_{1,2} & \cdots & 0\cdot a_{1,n} \\ & * & \end{vmatrix} + 0 \cdot * \right) \\
&= a_{1,1} + \frac{d}{dt}\bigg|_{t=0} \det(I_{2\times n, 2\times n} + tA_{2\times n, 2\times n}) \\
&= \cdots = a_{1,1} + \cdots + a_{n,n} = \operatorname{tr}(A),
\end{aligned}
$$

where we have developed the first column in $(*)$ and applied the product rule in $(**)$.

4. Let M be a smooth n-dimensional manifold and $p \in M$. Show that if (U, φ) is any chart at p with $\varphi(p) = 0$, then the map

$$\mathbb{R}^n \to T_pM, \quad v \mapsto (f \mapsto D_0(f \circ \varphi^{-1})(v))$$

is a vector space isomorphism.

5. Let M be a smooth manifold and $p \in M$. Show that for all open sets $U \subseteq M$ with $p \in U$ and every $g \in \mathcal{C}^\infty(U)$, there is $f \in \mathcal{C}^\infty(M)$ such that (U, g) and (M, f) define the same germ at p.

6. We have seen in class that if $\varphi\colon M \to M'$ is a diffeomorphism, then $\operatorname{Vect}^\infty(M) \to \operatorname{Vect}^\infty(M')$, $X \mapsto \varphi_*(X)$ is a Lie algebra isomorphism. Give a formula for $\varphi_*(X)$ in terms of derivatives of φ.

   Solution: See exercise class 2.