Solution 5

1. Compute the Lie algebra of $O(p,q)$ and $\mathrm{SO}(p,q)$ for $n=p+q$.

   Solution: Denote by $n := p+q$. Recall that the definition of the group $O(p,q)$ is given by
$$O(p,q) = \{X \in GL(n,\mathbb{R}) | {}^tX I_{p,q} X = I_{p,q}\}.$$

   To compute the Lie algebra associated to $O(p,q)$ we are going to realize this group as a fiber of a suitable constant rank map. We define
$$F : GL(n,\mathbb{R}) \to GL(n,\mathbb{R}), \quad F(X) := {}^tX I_{p,q} X.$$

   Clearly F is smooth since it can be expressed as a polynomial functions of the coordinates of the matrix X. Additionally, by definition we have $F^{-1}(I_{p,q}) = O(p,q)$. We are going to prove that the map F has constant rank. Let X be any element in $GL(n,\mathbb{R})$ and let Y be any tangent vector at X (that means $Y \in M(n,\mathbb{R})$). Using the usual definition of the differential in terms of smooth curves we have

$$\begin{aligned}(D_XF)(Y) =& \frac{d}{ds}|_{s=0} F(X+sY) = \frac{d}{ds}|_{s=0}({}^t(X+sY)I_{p,q}(X+sY)) = \\ =& \frac{d}{ds}|_{s=0}({}^tXI_{p,q}X + s({}^tXI_{p,q}Y + {}^tYI_{p,q}X) + s^2({}^tYI_{p,q}Y)) = \\ =& ({}^tXI_{p,q}Y + {}^tYI_{p,q}X) = {}^tX(I_{p,q}YX^{-1} + {}^t(X^{-1}){}^tYI_{p,q})X = \\ =& {}^tXD_{\mathrm{Id}}(YX^{-1})X.\end{aligned}$$

   From the equation above we deduce that the rank of F is constant and the Lie algebra of $O(p,q)$ is given by
$$\mathfrak{o}(p,q) = \mathrm{Lie}(O(p,q)) = \ker(D_{\mathrm{Id}}F) = \{X \in M(n,\mathbb{R}) | {}^tXI_{p,q} + I_{p,q}X = 0\}.$$

2. Realize $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$ and $U(n)$ as Lie groups, and compute their Lie algebras.

   Solution: See exercise class 2.

3. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Show that the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with the bracket
$$[(x_1,y_1),(x_2,y_2)] = ([x_1,x_2]_{\mathfrak{g}}, [y_1,y_2]_{\mathfrak{h}}).$$

Solution: We are going to denote by $X(M) = \mathrm{Vect}^\infty(M)$ the set of vector fields over a generic manifold M.

We are going to denote by

$$i_G : G \to G \times H, \quad i_G(g) := (g, e)$$

and similarly

$$i_H : H \to G \times H, \quad i_H(h) := (e, h).$$

In the same way, the differential of both maps induces inclusions

$$D_e i_G : T_eG \to T_eG \times T_eH, \quad D_e i_G(u) := (u, 0)$$

and

$$D_e i_H : T_eH \to T_eG \times T_eH, \quad D_e i_G(v) := (0, v).$$

Recall that $T_eG \times T_eH$ is canonically isomorphic to $T_eG \oplus T_eH$ as $\mathbb{R}$-vector spaces via the map which sends (u, v) to $u + v$, for every $u \in T_eG$ and every $v \in T_eH$. (In this way we get $D_e i_G$ is simply the inclusion of T_eG into $T_eG \oplus T_eH$ and the same for $D_e i_H$). This means that every element w in $T_{(e,e)}(G \times H)$ can be written uniquely as $w = u+v$, where $u \in T_eG$ and $v \in T_eH$, or equivalently we can identify $T_{(e,e)}(G \times H)$ with $T_eG \oplus T_eH$.

From the lecture, we know that there is a bijection between left-invariant vector fields on G (resp. H) and vectors of the tangent space T_eG (resp. T_eH) and the isomorphism is given by

$$L_G : T_eG \to X(G)^G, \quad L_G(u) := u^L$$

where the vector field u^L is defined at the point $g \in G$ as $u^L_g := D_e L_g(u)$.

It should be clear that we have the following commutative diagram

$$\begin{array}{ccc} T_eG \oplus T_eH & \xrightarrow{\cong} & T_{(e,e)}(G \times H) \\ \Big\downarrow{\scriptstyle L_G \oplus L_H} & & \Big\downarrow{\scriptstyle L_{G\times H}} \\ X(G)^G \oplus X(H)^H & \xrightarrow{\cong} & X(G \times H)^{G \times H}. \end{array}$$

The diagram above is telling us that every $(G \times H)$-left-invariant vector field $Z = w^L$, where $w \in T_{(e,e)}(G \times H)$, can be uniquely written as $Z = X + Y$, where $X = u^L$ (resp. $Y = v^L$) where $u \in T_eG$ (resp. $v \in T_eH$). Here the left-invariance property has to be understood in $G \times H$ (that means that both u^L and v^L are $G \times H$ left-invariant).

Take now $Z_1, Z_2 \in X(G \times H)^{G \times H}$ of the form $Z_i = w_i^L$, where $w_i \in T_{(e,e)}(G \times H)$ for $i = 1, 2$. By what we have said so far there exist unique $u_i \in T_eG$ and $v_i \in T_eH$ such that $w_i^L = u_i^L + v_i^L$, for $i = 1, 2$. It holds

$$
\begin{aligned}
[Z_1, Z_2] =& [w_1^L, w_2^L] = [u_1^L + v_1^L, u_2^L + v_2^L] = \\
=& [u_1^L, u_2^L] + [u_1^L, v_2^L] + [v_1^L, u_2^L] + [v_1^L, v_2^L].
\end{aligned}
$$

It is immediate to verify that for every $[u^L, v^L] = 0$ for any $u \in T_eG$ and $v \in T_eH$, hence we get

$$[w_1^L, w_2^L] = [u_1^L, u_2^L] + [v_1^L, v_2^L],$$

which is exactly the Lie algebra structure given on the product, and we are done.

4. Show that the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ denotes the cross product, is isomorphic to the Lie algebra of $O(3, \mathbb{R})$.

   Solution: Check that the map

$$\mathfrak{o}(3, \mathbb{R}) \to \mathbb{R}^3, \qquad \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   is a Lie algebra isomorphism.

5. Read and understand the pages from Boothby's book (see website) that give a complete proof of Proposition 3.43 in the notes.