Solution 6

1. Use Proposition 3.38 (3) to show that $\exp_{\mathrm{GL}(n,\mathbb{R})}(tA) = \mathrm{Exp}(tA)$ for all $t \in \mathbb{R}$ and $A \in \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}_{n,n}(\mathbb{R})$, where Exp denotes the matrix exponential.

   Solution: See proof of Corollary 3.8 in *From topological groups to Lie groups.*

2. Show that $\mathrm{Exp}\colon \mathfrak{u}(n) \to U(n)$ is surjective.

   Hint: Combine the fact that every $A \in U(n)$ is diagonalizable with the formula $g\mathrm{Exp}(X)g^{-1} = \mathrm{Exp}(gXg^{-1})$, which is valid for all $X \in \mathrm{Mat}_{n,n}(\mathbb{C})$ and $g \in \mathrm{GL}(n, \mathbb{C})$.

3. Show that $\mathrm{Exp}\colon \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is surjective.

   Hint: Use a similar argument as in Exercise 2 and the Jordan normal form.

4. Let V be a finite dimensional real vector space and $\Gamma < V$ a discrete subgroup. Show that there exist $\gamma_1, \ldots, \gamma_r \in \Gamma$, linearly independent in V such that $\Gamma = \mathbb{Z}\gamma_1 + \ldots + \mathbb{Z}\gamma_r$.

   Solution: We will prove this by induction on the dimension n.

   Let $n = 1$ and let $D < \mathbb{R}$ be a discrete subgroup. Without loss of generality we may assume that $D \neq \{0\}$. Since D is discrete there is $x_1 \in D \smallsetminus \{0\}$ such that $|x_1| = \min\{|x| : x \in D \smallsetminus \{0\}\}$. We claim that $D = \mathbb{Z}x_1$. Suppose there is $y \in D \smallsetminus \mathbb{Z}x_1$. Then there is $k \in \mathbb{Z}$ such that

   $$k \cdot x_1 < y < (k+1) \cdot x_1.$$

   It follows that $y - k \cdot x_1 \in D$ and $|y - k \cdot x_1| < |x_1|$ which contradicts the minimality of x_1. This shows that $D = \mathbb{Z}x_1$ and finishes the proof of the base case $n = 1$.

   Let $n \in \mathbb{N}$ and assume the statement holds for all discrete subgroups of $\mathbb{R}^{n-1}$. Let $D < \mathbb{R}^n$ be a discrete subgroup. Without loss of generality we may assume that $D \neq \{0\}$. There is $x_1 \in D \smallsetminus \{0\}$ such that $\|x_1\| = \min\{\|x\| : x \in D \smallsetminus \{0\}\}$. Consider the quotient $\mathbb{R}^n/\mathbb{R}x_1 \cong \mathbb{R}^{n-1}$ and the projection

   $$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{R} \cdot x_1 \cong \mathbb{R}^{n-1}$$

   onto it.

   We claim that $D' = \pi(D) < \mathbb{R}^{n-1}$ is a discrete subgroup. We will see this by showing that $V' := \pi(B_r(0))$ is an open neighborhood of $0 \in D'$ such that $V' \cap D' = \{0\}$ where $r := \inf\{\|t \cdot x_1 - y\| : t \in \mathbb{R}, y \in D \smallsetminus \mathbb{Z}x_1\}$.

   First of all, we need to see that r is in fact positive. In order to prove this let us verify that

   $$r = \inf\{\|t \cdot x_1 - y\| : t \in \mathbb{R}, y \in D \smallsetminus \mathbb{Z}x_1\} = \inf\{\|t \cdot x_1 - y\| : t \in [0,1], y \in D \smallsetminus \mathbb{Z}x_1\}.$$

Clearly, the left-hand-side is less than or equal to the right-hand-side. On the other hand, if $R \geqslant 0$ such that there are $t \in \mathbb{R}$ and $y \in D \smallsetminus \mathbb{Z}x_1$ satisfying $R \geqslant \|t \cdot x_1 - y\|$ then also

$$R \geqslant \|t \cdot x_1 - y\| = \|(t - \lfloor t \rfloor)x_1 - (y - \lfloor t \rfloor x_1)\|;$$

whence there are $s := t - \lfloor t \rfloor \in [0,1]$ and $w := (y - \lfloor t \rfloor x_1) \in D \smallsetminus \mathbb{Z}x_1$ such that $R \geqslant \|s \cdot x_1 - w\|$. Therefore, the right-hand-side is also less than or equal to the left-hand-side such that they must be equal. Because $\{t \cdot x_1 : t \in [0,1]\} \subset \mathbb{R}^n$ is compact and $D \smallsetminus \mathbb{Z}x_1$ is discrete the infimum on the right-hand-side is in fact a minimum. It is attained at some $t_0 \cdot x_1$ and $y_0 \in D \smallsetminus \mathbb{Z}x_1$. If $r = \|t_0 \cdot x_1 - y_0\| = 0$ then $y_0 = t_0 x_1$ and $t_0 \in (0,1)$ because $y_0 \notin \mathbb{Z}x_1$. But then $\|y_0\| = t_0\|x_1\| < \|x_1\|$ which contradicts the minimality of $\|x_1\|$; whence $r > 0$.

Clearly, $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is an open map such that $V' = \pi(B_r(0))$ is an open neighborhood of $0 \in \mathbb{R}^{n-1}$. Now, let $x' \in D' \cap V'$, i.e. $x' = \pi(u) = \pi(y)$ for some $u \in B_r(0)$, $y \in D$. Then $y - u \in \mathbb{R}x_1$, i.e. $y - u = t \cdot x_1$ for some $t \in \mathbb{R}$. This implies that

$$\|y - t \cdot x_1\| = \|u\| < r = \inf\{\|y - t \cdot x_1\| : t \in \mathbb{R}, y \in D \smallsetminus \mathbb{Z}x_1\}.$$

We deduce that $y \in \mathbb{Z}x_1 \subset \mathbb{R}x_1$; whence $x' = \pi(y) = 0$ and $V' \cap D' = \{0\}$. Therefore, 0 is an isolated point in D' such that D' is a discrete subgroup of $\mathbb{R}^{n-1}$ as claimed.

By the induction hypothesis there are $x'_2, \ldots, x'_k \in D' < \mathbb{R}^{n-1}$ which are linearly independent over $\mathbb{R}$ and generate D' as a $\mathbb{Z}$-submodule, i.e. $D' = \mathbb{Z}x'_2 \oplus \cdots \oplus \mathbb{Z}x'_k$. We choose for every x'_i a preimage $x_i \in \pi^{-1}(x'_i) \cap D$. These $x_1, x_2, \ldots, x_k \in D$ are linearly independent over $\mathbb{R}$ and satisfy $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$. Indeed, let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k = 0. \tag{1}$$

Then

$$\begin{aligned} 0 &= \pi(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) \\ &= \underbrace{\lambda_1 \pi(x_1)}_{=0} + \lambda_2 \pi(x_2) + \cdots + \lambda_k \pi(x_k) \\ &= \lambda_2 x'_2 + \cdots + \lambda_k x'_k. \end{aligned}$$

Because $x'_2, \ldots, x'_k$ are linearly independent, $\lambda'_2 = \ldots = \lambda'_k = 0$. By (1), $\lambda_1 x_1 = 0$. Finally, since $x_1 \neq 0$ also $\lambda_1 = 0$.

In order to see that $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-module, let $y \in D$. Then

$$\pi(y) = a_2 x'_2 + \cdots + a_k x'_k = a_2 \pi(x_2) + \cdots + a_k \pi(x_k)$$

for some $a_2, \ldots, a_k \in \mathbb{Z}$ since $x'_2, \ldots, x'_k$ generate D' as a $\mathbb{Z}$-module. Considering $y' = a_2 x_2 + \cdots + a_k x_k \in D$ we obtain

$$\pi(y') = \pi(a_2 x_2 + \cdots + a_k x_k) = a_2 \pi(x_2) + \cdots + a_k \pi(x_k) = \pi(y)$$

by linearity such that $y - y' \in D \cap \ker \pi = D \cap \mathbb{R}x_1$.

We claim that $D \cap \ker \pi = \mathbb{Z}x_1$. It is immediate that $\mathbb{Z}x_1 \subseteq D \cap \ker \pi$. To see the other inclusion suppose that there is $t \cdot x_1 \in D$ for some $t \in \mathbb{R} \setminus \mathbb{Z}$. Then $w = (t - \lfloor t \rfloor) \cdot x_1 \in D \setminus \{0\}$ and

$$\|w\| = (t - \lfloor t \rfloor) \cdot \|x_1\| < \|x_1\|$$

in contradiction to the minimality of x_1.

Therefore, $y - y' \in \mathbb{Z}x_1$ and there exists $a_1 \in \mathbb{Z}$ such that

$$y = a_1 x_1 + y' = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k.$$

Hence, $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$.

5. Show that every connected abelian Lie group G is isomorphic as Lie groups to $\mathbb{T}^a \times \mathbb{R}^{n-a}$ for some $a \in \{0, \dots, n\}$, where $n = \dim G$ and $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.

   Solution: See proof of Corollary 3.7 in *From topological groups to Lie groups.*

6. Let G be a Lie group. Show that there is an open neighborhood of e which does not contain any non-trivial subgroup of G.

   Solution: See proof of Theorem 3.12 in *From topological groups to Lie groups.*