Solution 7

1. Review the proof of Cartan's Theorem.

2. Let $H < G$ be a closed subgroup of a Lie group G with Lie algebra $\mathfrak{g}$. Show that

$$\mathrm{Lie}(H) = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \,\forall t \in \mathbb{R}\}.$$

Solution: In the proof of Cartan's theorem we have seen that

$$\begin{aligned} W := \{0\} \cup \{X \in \mathfrak{g} \smallsetminus (0) : \exists (X_n) \in \mathfrak{g} \smallsetminus (0) \text{ such that} \\ \exp_G(X_n) \in H \,\forall n \geqslant 1,\ \lim_{n\to\infty} X_n = 0,\ \lim_{n\to\infty} \tfrac{X_n}{\|X_n\|} = \tfrac{X}{\|X\|}\} \end{aligned}$$

can be identified with the tangent space at e of H. Thus it suffices to show that $W = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \,\forall t \in \mathbb{R}\}$.

Let thus $X \in W$ and assume $X \neq 0$. By (1) in the proof of Cartan's theorem we have seen that $\exp_G(W) \subseteq H$. Since $tX \in W$ for all $t \in \mathbb{R}$, we have $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$, which shows the first inclusion.

On the other hand let $X \in \mathfrak{g}$ such that $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$. Assume $X \neq 0$, and set $X_n := \frac{1}{n}X$ for all $n \geqslant 1$. Then $X_n \in \mathfrak{g} \smallsetminus (0)$, $\lim_{n\to\infty} X_n = \lim_{n\to\infty} \frac{1}{n}X = 0$ and

$$\lim_{n\to\infty} \tfrac{X_n}{\|X_n\|} = \lim_{n\to\infty} \tfrac{1/nX}{\|1/nX\|} = \tfrac{X}{\|X\|},$$

so $X \in W$.

3. Show that a continuous group homomorphism between two Lie groups is smooth.

Hint: Look at the graph of the map and apply Cartan's theorem.

Solution: Let $\varphi \colon G \to H$ be a continuous group homomorphism of Lie groups. Then $\mathrm{Graph}(\varphi) = \{(g, \varphi(g)) : g \in G\} \subseteq G \times H$ is a closed subgroup of a Lie group, hence by Cartan's theorem a Lie group. Consider now the map $\Gamma_\varphi \colon G \to \mathrm{Graph}(\varphi)$, $g \mapsto (g, \varphi(g))$, which is a homeomorphism of groups and whose inverse is the restriction of the projection $G \times H \to G$ to $\mathrm{Graph}(\varphi)$. The inverse of Γ_φ is smooth with constant rank, and hence Γ_φ is a diffeomorphism. If now q denotes the projection $G \times H \to H$ on the second factor, then $\varphi = q \circ \Gamma_\varphi$. Since both q and Γ_φ are smooth, so is φ.

4. Read the pages 32-34 in *Representations of Compact Lie Groups* by Bröcker-Dieck.

5. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}(g)$ is smooth, where $\mathrm{Ad}(g) := D_e(\mathrm{int}(g))$.

   Hint: Apply Proposition 3.50 to the map $\mathrm{int}(g)\colon G \to G$, $x \mapsto gxg^{-1}$. Use that $\exp_G$ is a local diffeomorphism to conclude that Ad is smooth near e. Then use left translation to show that Ad is smooth everywhere.

   Solution: Consider the map $F\colon G \times G \to G$ defined by $F(g,h) := ghg^{-1}$. This is smooth, so its differential $DF\colon TG \times TG \to TG$ is smooth. Restrict in the second component to the submanifold $T_eG = \mathfrak{g}$. The zero vector field $0\colon G \to TG$ is a smooth map, thus the map

   $$G \times \mathfrak{g} \to TG,\ (g,X) \mapsto DF(0(g),X)$$

   is smooth as well. From the construction, we have

   $$D_{(g,e)}F(0(g),X) = \tfrac{d}{dt}_{|t=0} F(g,\exp(tX)) = \tfrac{d}{dt}_{|t=0} g\exp(tX)g^{-1} = \mathrm{Ad}(g)(X) \in T_eG.$$

   Thus the map $G \times \mathfrak{g} \to \mathfrak{g}$, $(g,X) \mapsto \mathrm{Ad}(g)(X)$ is smooth. If you choose a basis for $\mathfrak{g}$, say $\{X_i\}$ with dual basis $\{X_i^*\}$, then the entries of the matrix representing $\mathrm{Ad}(g)$ with respect to the basis $\{X_i\}$ is $X_i^*(\mathrm{Ad}(g)(X_j))$, so they depend smoothly on g, thus $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is smooth.

6. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \lhd \mathfrak{g}$ an abelian ideal in $\mathfrak{g}$. Show that $\exp_G(\mathfrak{a})$ is a normal subgroup of G.

   Solution: Since $\mathfrak{a}$ is abelian $\exp_{G|\mathfrak{a}}$ is a homomorphism, and $A := \exp_G(\mathfrak{a})$ is a subgroup of G. Since G is connected it suffices to prove the claim for elements in a neighborhood U of e. We can take this neighborhood such $\exp_G\colon \mathfrak{g} \to G$ is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto it. Thus for all $g \in U$ there exists $Y \in \mathfrak{g}$ such that $g = \exp_G(Y)$. We thus have using the naturality of $\exp_G$ for all $X \in \mathfrak{a}$

   $$\begin{aligned} \exp_G(Y)\exp_G(X)\exp_G(Y)^{-1} &= \mathrm{int}(\exp_G(Y)) \circ \exp_G(X) \\ &= \exp_G(\mathrm{Ad}(\exp_G(Y))X) \\ &= \exp_G(\mathrm{Exp}(\mathrm{ad}(Y))X) \\ &= \exp_G\left(\sum_{k=0}^{\infty} \tfrac{1}{k!}\mathrm{ad}(Y)^k X\right) \\ &\in \exp_G(\mathfrak{a}), \end{aligned}$$

   since $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. Thus $\exp_G(\mathfrak{a})$ is normal in G.

7. Let G be a topological group and $H < G$ a closed subgroup. Show that if H and G/H are connected, then so is G.

   Solution: We suppose that H and G/H are connected and that $G = A \cup B$ for disjoint, non-empty open sets A and B in G. Assume without loss of generality that $e \in A$. Since H is connected, all of its left cosets $gH = L_g(H)$ are. Thus since each coset meets either A or B it must be contained entirely in one of the two. Consequently, A and B are union of left cosets of H. If now $p\colon G \to G/H$ denotes the projection map on left cosets, it follows that both $p(A)$ and $p(B)$ are non-empty disjoint. Since p is open, $p(A)$ and $p(B)$ are open non-empty disjoint whose union is G/H, which contradicts the connectedness of G/H.