Solution 8

1. Convince yourself that all the definitions and results in Chapter 4 concerning uniquely real Lie algebras go over without modifications to complex Lie algebras.

2. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{g}$ as a vector space. Show that the bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ extends uniquely to a $\mathbb{C}$-bilinear map $[\cdot,\cdot]_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ turning $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra.

3. In the setting of Exercise 2 show that the canonical injection $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$, $X \mapsto X \otimes 1$ is a homomorphism of real Lie algebras and, if we identify $\mathfrak{g}$ with its image in $\mathfrak{g}_{\mathbb{C}}$, we have that
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}.$$
Express the bracket of $\mathfrak{g}_{\mathbb{C}}$ in this decomposition.

   Solution: We have $[X + iY, X' + iY']_{\mathbb{C}} = [X, X'] - [Y, Y'] + i\left([Y, X'] + [X, Y']\right)$ for all $X + iX', Y + iY' \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$.

4. In the setting of Exercises 2 and 3 show that
   (a) $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{\mathbb{C}}$ is solvable,
   (b) $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}_{\mathbb{C}}$ is nilpotent.

   Solution:

   (a) Similar to (b).
   (b) We note that $\mathfrak{g}$ is a real Lie subalgebra of the real Lie algebra $\mathfrak{g}_{\mathbb{C}}$, hence nilpotent if $\mathfrak{g}_{\mathbb{C}}$ is.
   We will show inductively that
$$C^n(\mathfrak{g}^{\mathbb{C}}) \subseteq C^n(\mathfrak{g}) + iC^n(\mathfrak{g})$$
   for all $n \in \mathbb{N}$. Clearly, this suffices to prove the forward direction.
   Base case $n = 1$:
$$\begin{aligned} C^1(\mathfrak{g}^{\mathbb{C}}) &= [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}] \\ &= [\mathfrak{g} + i\mathfrak{g}, \mathfrak{g} + i \cdot \mathfrak{g}] \\ &= [\mathfrak{g}, \mathfrak{g}] + i[\mathfrak{g}, \mathfrak{g}] + i[\mathfrak{g}, \mathfrak{g}] - [\mathfrak{g}, \mathfrak{g}] \\ &\subseteq C^1(\mathfrak{g}) + iC^1(\mathfrak{g}) \end{aligned}$$
   Inductive step:

$$
\begin{aligned}
C^{n+1}(\mathfrak{g}^{\mathbb{C}}) &= [C^n(\mathfrak{g}^{\mathbb{C}}), \mathfrak{g}^{\mathbb{C}}] \\
&\subseteq [C^n(\mathfrak{g}) + iC^n(\mathfrak{g}), \mathfrak{g} + i\mathfrak{g}] \\
&= [C^n(\mathfrak{g}), \mathfrak{g}] + i[C^n(\mathfrak{g}), \mathfrak{g}] + i[C^n(\mathfrak{g}), \mathfrak{g}] - [C^n(\mathfrak{g}), \mathfrak{g}] \\
&\subseteq C^{n+1}(\mathfrak{g}) + iC^{n+1}(\mathfrak{g})
\end{aligned}
$$