Solution 9

1. Read pages 4-37 to 4-39 and verify that everything works as stated.

2. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and Killing form $K_\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and $g \in G$
$$K_\mathfrak{g}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = K_\mathfrak{g}(X, Y),$$
where $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G.

   Hint: Compute the derivative of $\Phi(t) := K_\mathfrak{g}(\mathrm{Ad}(\exp tZ)X, \mathrm{Ad}(\exp tZ)Y)$ and use Proposition 4.46.

   Solution: Show that in a neighborhood U of e in G, the derivative is constant equal to 0. For this use that if B is a bilinear form on a vector space then $D_{(x,y)}B(v,w) = B(x,w) + B(v,y)$. Furthermore, remark that $\mathrm{Ad} \circ \exp = \mathrm{Exp} \circ \mathrm{ad}$, and we understand how to differentiate Exp. Since $K_\mathfrak{g}(\mathrm{Ad}(g)\,\cdot, \mathrm{Ad}(g)\,\cdot)$ and $K_\mathfrak{g}(\cdot,\cdot)$ agree at $g = e \in U$, they agree on all of U. Now G connected implies the claim since U generates G.

3. Let $\mathfrak{g}$ be a real Lie algebra.

   (a) Show that under the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ we have that
   $$K_\mathfrak{g} = K_{\mathfrak{g}_\mathbb{C}|_{\mathfrak{g}\times\mathfrak{g}}}.$$

   (b) Show that $K_{\mathfrak{g}_\mathbb{C}|_{\mathfrak{g}_\mathbb{C}^{(1)} \times \mathfrak{g}_\mathbb{C}^{(1)}}} = 0$ if and only if $K_{\mathfrak{g}|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}}} = 0$.

   Hint: Use that $\mathfrak{g}_\mathbb{C}^{(1)} = \mathfrak{g}^{(1)} + i\mathfrak{g}^{(1)}$.

   Hint:

   (a) Use the same decomposition as in class.

   (b) Another hint would be to use Cartan's solvability criterion and Exercise 3 (a) on Exercise Sheet 8.

4. Compute the Killing form of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. *Solution*: See exercise class.

5. Consider the three-dimensional Heisenberg group
$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that the center of H is

$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

Let $D < Z(H)$ be the following discrete subgroup

$$D := \mathrm{SL}_3(\mathbb{Z}) \cap Z(H) = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Check that $G := H/D$ is a connected, solvable Lie group and show that G does not admit a smooth, injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space V.

Solution: Since H is connected, so is G, and since both H and D are solvable, so is G. Assume $\pi\colon G \to \mathrm{GL}(V)$ is a smooth homomorphism for a finite-dimensional $\mathbb{C}$-vector space V. We will show that $\pi(Z(H)/D) = \mathrm{id}$, that is $Z(H)/D < \ker \pi$, so that π cannot be injective.

Let us observe first of all that, since D is discrete, then

$$\mathrm{Lie}(H/D) = \mathrm{Lie}(H) = \mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subset \mathfrak{gl}(3, \mathbb{R})$$

and it is moreover solvable. Furthermore

$$\mathrm{Lie}(Z(H)/D) = \mathrm{Lie}(Z(H)) = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = [\mathfrak{h}, \mathfrak{h}] \,.$$

By Lie's theorem, if $\rho := d_e\pi$, the image $\rho(\mathfrak{h})$ is upper triangular, so that $[\rho(\mathfrak{h}), \rho(\mathfrak{h})]$ is strictly upper triangular. Thus

$$\rho(\mathrm{Lie}(Z(H)/D)) = \rho([\mathfrak{h}, \mathfrak{h}]) = [\rho(\mathfrak{h}), \rho(\mathfrak{h})] \subset \left\{ \begin{pmatrix} 0 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 0 \end{pmatrix} \right\},$$

from which it follows that

$$\pi(Z(H)/D) < \left\{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\} =: L \,.$$

Observe that since $Z(H)/D \simeq S^1$, then $\pi(Z(H)/D) =: K$ is a compact subgroup of L. We will show now that L cannot have non-trivial compact subgroups, which forces $K = \mathrm{id}$. In order to show this, we will show that any compact subgroup can be conjugated into any small neighborhood of $\mathrm{id} \in \mathrm{GL}(n, \mathbb{C})$, thus contradicting that L is a Lie group.

To this purpose, let $g = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathrm{GL}(n, \mathbb{C})$ a diagonal matrix with entries $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$. Then, if $i < j$,

$$\begin{aligned}\left(c_g\begin{pmatrix}1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1\end{pmatrix}\right)_{ij} &= \left(\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\begin{pmatrix}1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1}\end{pmatrix}\right) \\ &= \frac{\lambda_i}{\lambda_j}\begin{pmatrix}1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1\end{pmatrix}_{ij},\end{aligned}$$

so that

$$\left(c_g^n\begin{pmatrix}1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1\end{pmatrix}\right)_{ij} = \left(\frac{\lambda_i}{\lambda_j}\right)^n\begin{pmatrix}1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1\end{pmatrix}_{ij} \tag{1}$$

If $\begin{pmatrix}1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1\end{pmatrix} \in K$ its entries are bounded and, since $\lambda_i/\lambda_j < 1$, the right hand side of (1) converges to id and is hence eventually contained in any neighborhood of id, no matter how small.

6. Show the following exceptional isomorphisms of Lie algebras.

   (a) Show that $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$. Hint: If $\dim V = 4$ then $\dim \Lambda^2 V = 6$.

   (b) Show that $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

   (c) Show that $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$.

   (d) Show that $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$.

   Solution:

(a) Consider the standard action of an element $g \in \mathrm{SL}(4, \mathbb{C})$ on the vector space $\mathbb{C}^4$, that means

$$g.v := gv, \quad v \in \mathbb{C}^4,$$

where gv is the standard multiplication rows-by-columns. This action determines an action on the space $\mathbb{C}^4 \otimes \mathbb{C}^4$ and hence on the space $\Lambda^2\mathbb{C}^4$ given by

$$g.(u \wedge v) := gu \wedge gv, \quad u, v \in \mathbb{C}^4.$$

In this way we obtain a morphism $\mathrm{SL}(4, \mathbb{C}) \to \mathrm{SL}(6, \mathbb{C})$. We need to show that its image actually is contained in $\mathrm{SO}(6, \mathbb{C})$.

The key point now is that $\Lambda^4\mathbb{C}^4 \cong \mathbb{C}$ and if we fix the canonical basis $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ a generator is given by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. This allows us to define a symmetric bilinear form $B \colon \Lambda^2\mathbb{C}^4 \times \Lambda^2\mathbb{C}^4 \to \mathbb{C}$ as it follows. Consider $u_1 \wedge v_1, u_2 \wedge v_2 \in \Lambda^2\mathbb{C}^4$. Since $\Lambda^4\mathbb{C}^4$ is generated by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$, we can define $B(u_1 \wedge v_1, u_2 \wedge v_2)$ to be the unique scalar for which it holds

$$u_1 \wedge v_1 \wedge u_2 \wedge v_2 = B(u_1 \wedge v_1, u_2 \wedge v_2) e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

The function B defined above is a symmetric bilinear form which is non-degenerate (you can express the associated matrix in the basis $\Lambda^2\mathcal{E} = \{e_i \wedge e_j\}_{i<j}$ and check that the determinant is different from zero). In addition the $\mathrm{SL}(4, \mathbb{C})$-action on $\Lambda^2\mathbb{C}^4$ preserves B (you can check it on the elements on the basis since B is bilinear). Thus the representation $\mathrm{SL}(4, \mathbb{C}) \to \mathrm{SL}(6, \mathbb{C})$ has image contained in $\mathrm{SO}(6, \mathbb{C})$, as desired. This is a smooth homomorphism whose kernel is equal to $\{\mathrm{Id}, -\mathrm{Id}\}$, hence it induces an isomorphism between the associated Lie algebras, as desired.

(b) Using the definition it is easy to verify that $\mathrm{SL}(2, \mathbb{C}) = \mathrm{Sp}(2, \mathbb{C})$ (see (d) below). This means that any $g \in \mathrm{SL}(2, \mathbb{C})$ preserves the standard symplectic form given by

$$\omega(u, v) := {}^t u J_2 v, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for every $u, v \in \mathbb{C}^2$.

By using the symplectic form ω we can define the following function

$$B \colon \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}, \quad B(u_1 \otimes v_1, u_2 \otimes v_2) = \omega(u_1, u_2)\omega(v_1, v_2).$$

The function B is a symmetric bilinear form which is non-degenerate. If we now consider the action of $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ given by $(g, h)(u \otimes v) := (gu \otimes hv)$ for every $g, h \in \mathrm{SL}(2, \mathbb{C}), u, v \in \mathbb{C}^2$, we get that this action preserves B. Hence we get a smooth homomorphism $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(4, \mathbb{C})$ whose kernel is $\{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\}$. The induced homorphism on the associated Lie algebras is the desired isomorphism.

(c) Recall that the Lie algebra

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in M(2,\mathbb{C}) | \mathrm{tr}(X) = 0\}$$

admits the following natural basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the same way the Lie algebra

$$\mathfrak{so}(3,\mathbb{C}) = \{X \in M(3,\mathbb{C}) |{}^t X + X = 0\}$$

has a natural basis given by $h_{ij} = E_{ij} - E_{ji}$ for $i, j = 1, 2, 3$ and $i < j$. Here E_{ij} denotes the matrix with 1 in the only entry of indices (i, j) and equal to zero elsewhere.

Define a map $\varphi \colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{C})$ in the following way

$$\varphi(H) = -2ih_{13}, \ \ \varphi(E) = ih_{12} + h_{23}, \ \ \varphi(F) = -ih_{12} + h_{23}. \tag{2}$$

It easy to verify that the map φ gives us the desired isomomorphism.

(d) It follows immediately by the definition that $\mathrm{Sp}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$ and from this it follows the isomorphism between the associated Lie algebras (they are the same).

We move now to $\mathrm{Sp}(4,\mathbb{C})$. Consider the standard action of $\mathrm{Sp}(4,\mathbb{C})$ on $\Lambda^2\mathbb{C}^4$ (the same defined in Exercise 1) and the same bilinear form B. Since $\mathrm{Sp}(4,\mathbb{C}) < \mathrm{SL}(4,\mathbb{C})$ we get that any $g \in \mathrm{Sp}(4,\mathbb{C})$ preserves B. If $\mathcal{E}$ denotes the canonical basis of $\mathbb{C}^4$, since g preserves the standard symplectic form on $\mathbb{C}^4$, it must fixes the element

$$\sigma := e_1 \wedge e_3 + e_2 \wedge e_4 \in \Lambda^2\mathbb{C}^4.$$

It easy to see that $B(\sigma, \sigma) \neq 0$. This means that the $\mathrm{Sp}(4,\mathbb{C})$-action preserves the decomposition

$$\Lambda^2\mathbb{C}^4 = \langle\sigma\rangle \oplus \langle\sigma\rangle^\perp$$

and also the restriction of B to $\langle\sigma\rangle^\perp$. This implies that we get a smooth homomorphism $\mathrm{Sp}(4,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C})$ which induces the desired isomorphism between the associated Lie algebras.

7. The goal of this exercise is to show that the spin group $\mathrm{SU}(2,\mathbb{C})$ is the universal covering group of the rotation group $\mathrm{SO}(3,\mathbb{R})$. Consider

$$\mathrm{SU}(2,\mathbb{C}) := \{g \in \mathrm{SL}(2,\mathbb{C}) : g^* g = I\} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \right\}$$

and its Lie algebra

$$\mathfrak{su}(2,\mathbb{C}) = \{X \in \mathfrak{sl}(2,\mathbb{C}) : X^* + X = 0\} = \left\{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} : a \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

(a) Construct a Lie group homomorphism $\varphi : \mathrm{SU}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{R})$ whose kernel is $\{\pm I\}$.
Hint: Use the adjoint representation of $\mathrm{SU}(2,\mathbb{C})$ on $\mathfrak{su}(2,\mathbb{C})$ and show that

$$b(X,Y) := -\frac{1}{2}\mathrm{tr}(XY)$$

defines a positive definite symmetric bilinear form on $\mathfrak{su}(2,\mathbb{C})$ considered as a real vector space.

(b) Show that $d_I\varphi : \mathfrak{su}(2,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{R})$ is a Lie algebra isomorphism and deduce that φ is a covering map.

(c) Show that $\mathrm{SU}(2,\mathbb{C})$ is homeomorphic to the 3-sphere $\mathbb{S}^3$ and deduce that $\mathrm{SU}(2,\mathbb{C})$ is simply connected. Show that $\mathrm{SO}(3,\mathbb{R})$ is homeomorphic to the three-dimensional real projective space $\mathbb{RP}^3$. What is the fundamental group of $\mathrm{SO}(3,\mathbb{R})$?

(d) Are there any other Lie groups whose Lie algebra is isomorphic to $\mathfrak{su}(2,\mathbb{C})$?
Hint: Analyze the discrete normal subgroups of $\mathrm{SU}(2,\mathbb{C})$.

Solution:

(a) Observe that a $\mathbb{R}$-basis for $\mathfrak{su}(2,\mathbb{C})$ is given by the matrices

$$U_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, U_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, U_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};$$

in particular $\dim_{\mathbb{R}} \mathfrak{su}(2,\mathbb{C}) = 3$ and $\mathfrak{su}(2,\mathbb{C}) \cong \mathbb{R}^3$ as vector spaces over $\mathbb{R}$. It is immediate that $b(\cdot,\cdot)$ as defined above is bilinear. Further, it is symmetric due to the fact that

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) \tag{3}$$

for all $X, Y \in \mathbb{R}^{n\times n}$. One computes directly that

$$b(U_i, U_j) = \delta_{ij}$$

for every $i, j = 1, 2, 3$ whence $b(\cdot,\cdot)$ corresponds to the standard Euclidean product under the vector space isomorphism $\mathfrak{su}(2,\mathbb{C}) \cong \mathbb{R}^3$.
As we have discussed in the lecture the adjoint representation $\mathrm{Ad}\colon \mathrm{SU}(2,\mathbb{C}) \to \mathrm{GL}(\mathfrak{su}(2,\mathbb{C}))$ is given by matrix conjugation

$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

for all $g \in \mathrm{SU}(2,\mathbb{C})$, $X \in \mathfrak{su}(2,\mathbb{C})$. We use this fact in order to see that b is in fact Ad-invariant:

$$\begin{aligned} b(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) &= -\frac{1}{2}\mathrm{tr}(gXg^{-1}gYg^{-1}) = -\frac{1}{2}\mathrm{tr}(gXYg^{-1}) \\ &\overset{(3)}{=} -\frac{1}{2}\mathrm{tr}(g^{-1}gXY) = -\frac{1}{2}\mathrm{tr}(XY) \\ &= b(X,Y) \end{aligned}$$

for all $X, Y \in \mathfrak{su}(2,\mathbb{C})$ and every $g \in \mathrm{SU}(2,\mathbb{C})$. In other words the adjoint action of $\mathrm{SU}(2,\mathbb{C})$ on the inner product space $(\mathfrak{su}(2,\mathbb{C}), b)$ is isometric and hence takes values in $O(\mathfrak{su}(2,\mathbb{C}), b) \cong O(3,\mathbb{R})$. By the first part of c) $\mathrm{SU}(2,\mathbb{C})$ is homeomorphic to the 3-sphere $\mathbb{S}^3$ and hence connected (there will be no cyclic reasoning!). Being a smooth homomorphism from $\mathrm{SU}(2,\mathbb{C})$ to $O(3,\mathbb{R})$ the adjoint representation sends $\mathrm{SU}(2,\mathbb{C})$ in the connected component of the identity of $O(3,\mathbb{R})$ which is $\mathrm{SO}(3,\mathbb{R}) = O(3,\mathbb{R})^\circ$. Therefore, the adjoint representation gives rise to a Lie group homomorphism

$$\mathrm{Ad} : \mathrm{SU}(2,\mathbb{C}) \longrightarrow O(\mathfrak{su}(2,\mathbb{C}), b)^\circ \cong \mathrm{SO}(3,\mathbb{R}).$$

Let us now prove that $\ker \mathrm{Ad} = \{\pm I\}$. Since $\pm I \in \mathrm{SU}(2,\mathbb{C})$ commutes with every matrix in $\mathrm{GL}(2,\mathbb{C})$ it is immediate that $\{\pm I\} \subseteq \ker \mathrm{Ad}$. In view of the other inclusion observe that

$$\begin{aligned} g \in \ker \mathrm{Ad} &\iff \mathrm{Ad}(g)(X) = X \qquad \forall X \in \mathfrak{su}(2,\mathbb{C}) \\ &\iff gU_i g^* = U_i \qquad \forall i = 1,2,3. \end{aligned}$$

Writing

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

we obtain

$$\begin{aligned} gU_1g^* &= \begin{pmatrix} bi\bar{a} + ai\bar{b} & ia^2 - ib^2 \\ i\,(\bar{a})^2 - i\,(\bar{b})^2 & b(-i)\bar{a} - ia\bar{b} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ gU_2g^* &= \begin{pmatrix} b\bar{a} - a\bar{b} & -a^2 - b^2 \\ (\bar{a})^2 + (\bar{b})^2 & a\bar{b} - b\bar{a} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ gU_3g^* &= \begin{pmatrix} ia\bar{a} - ib\bar{b} & -2iab \\ -2i\bar{a}\bar{b} & ib\bar{b} - ia\bar{a} \end{pmatrix} \overset{!}{=} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned}$$

These equations then easily imply $a = \pm 1, b = 0$, i.e. $\ker \mathrm{Ad} \subseteq \{\pm I\}$.

(b) As discussed in the lecture we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{su}(2,\mathbb{C}) & \xrightarrow{d_e\mathrm{Ad}=\mathrm{ad}} & \mathfrak{so}(3,\mathbb{R}) \\ \big\downarrow{\scriptstyle \mathrm{Exp}} & & \big\downarrow{\scriptstyle \mathrm{Exp}} \\ \mathrm{SU}(2,\mathbb{C}) & \xrightarrow{\mathrm{Ad}} & \mathrm{SO}(3,\mathbb{R}) \end{array}$$

Further,

$$\mathfrak{so}(3,\mathbb{R}) = \{X \in \mathfrak{sl}(2,\mathbb{R}) : X^T = -X\}$$

and $\dim_{\mathbb{R}} \mathfrak{so}(3,\mathbb{R}) = 3 = \dim_{\mathbb{R}} \mathfrak{su}(2,\mathbb{C})$. Hence it suffices to show that Ad is injective in order to see that $d_e\mathrm{Ad}$ is a Lie algebra isomorphism.

Let $X \in \ker \mathrm{ad}$. Then

$$0 = \mathrm{ad}(X)(Y) = [X, Y] = XY - YX$$

for all $Y \in \mathfrak{su}(2, \mathbb{C})$; or equivalently

$$XU_1 - U_1 X = 0, \quad XU_2 - U_2 X = 0, \quad XU_3 - U_3 X = 0.$$

A direct computation yields

$$\begin{aligned} XU_1 - U_1 X &= \begin{pmatrix} -i\,(z + z^*) & -2a \\ 2a & i\,(z + z^*) \end{pmatrix}, \\ XU_2 - U_2 X &= \begin{pmatrix} z - z^* & -2ia \\ -2ia & z^* - z \end{pmatrix}, \\ XU_3 - U_3 X &= \begin{pmatrix} 0 & 2iz^* \\ 2iz & 0 \end{pmatrix}, \end{aligned}$$

which easily implies $a = z = 0$ whence $\ker \mathrm{ad} = \{0\}$. Therefore, ad is injective and a Lie algebra isomorphism. We claim that this implies that $\mathrm{Ad} : \mathrm{SU}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{R})$ is a covering map. Let us for this recall a more general statement, which immediately implies the above.

Claim: Let H, G be arbitrary Lie groups and let G be connected. Further, let $\varphi\colon H \to G$ be a Lie group homomorphism. Show that φ is a covering map if and only if $D_e\varphi : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism.

Proof of claim: First suppose that φ is a covering map. Note that $D_{\tilde{e}}\varphi$ is a Lie algebra homomorphism since φ is a smooth homomorphism. Because φ is additionally a smooth covering map there are open neighborhoods $U \subseteq G$ of e and $V \subseteq H$ of $\tilde{e}$ such that $\varphi|_V\colon V \to U$ is a diffeomorphism. In particular, $D_e\varphi\colon T_{\tilde{e}}H \cong \mathfrak{h} \to T_eG \cong \mathfrak{g}$ is bijective such that $D_{\tilde{e}}\varphi$ is a Lie algebra isomorphism. By a lemma from the lecture every local homomorphism with bijective $D_{\tilde{e}}\varphi\colon \mathfrak{h} \to \mathfrak{g}$ is a local isomorphism.

Now, assume that $\varphi : H \to G$ is a smooth homomorphism such that $D_e\varphi\colon \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. This means that $D_{\tilde{e}}\varphi : T_{\tilde{e}}H \to T_eG$ is invertible such that by the inverse function theorem there are open neighborhoodes $U \subseteq G$ about $e \in G$ and $V \subseteq H$ about $\tilde{e} \in H$ such that $\varphi|_V\colon V \to U$ is a diffeomorphism. Because G is connected the open neighborhood U about $e \in G$ generates G and it follows easily that $\varphi\colon H \to G$ is surjective.

Now, choose a symmetric open neighborhood $W \subseteq V$ about $\tilde{e} \in H$ such that $W^2 \subseteq V$. Consider the open subset $U' := \varphi(W) \subseteq U$. We claim that $\varphi^{-1}(U') = \bigsqcup_{h \in \ker\varphi} Wh$ and $\varphi|_{Wh}\colon Wh \to U'$ is a diffeomorphism for all $h \in \ker\varphi$. Because $h \in \ker\varphi$ we have that $\varphi \circ R_h = \varphi$. Further $\varphi\colon W \to U'$ is a diffeomorphism such that also $\varphi\colon Wh \to U'$ is a diffeomorphism. Also,

$$
\begin{aligned}
&x \in \varphi^{-1}(U') = \varphi^{-1}(\varphi(W)) \iff \varphi(x) \in \varphi(W) \\
\iff& \exists w \in W : \varphi(x) = \varphi(w) \iff \exists w \in W : \varphi(w^{-1}x) = e \\
\iff& \exists w \in W : w^{-1}x \in \ker\varphi \iff x \in \bigcup_{h \in \ker\varphi} Wh,
\end{aligned}
$$

such that $\varphi^{-1}(U') = \bigcup_{h\in\ker\varphi} Wh$. Finally, if $Wh \cap Wh' \neq \varnothing$ for some $h, h' \in \ker\varphi$ then there are $w, w' \in W$ such that $wh = w'h'$, i.e. $h^{-1}h' \in W^2 \subseteq V$. Because $\varphi|_V \colon V \to U$ is injective and also $\varphi(h^{-1}h') = \varphi(h^{-1})\varphi(h') = e$ it follows that $h^{-1}h' = \tilde{e}$, or equivalently $h = h'$. Thus, $\bigcup_{h\in\ker\varphi} Wh$ is a disjoint union as claimed.

Using this together with the fact that φ is a homomorphism proves that φ is a covering map.

(c) In order to show that $\mathrm{SU}(2,\mathbb{C})$ is homeomorphic to $\mathbb{S}^3$ we consider its natural action on $\mathbb{C}^2$. Let us equip $\mathbb{C}^2$ with the standard hermitian inner product

$$
(z_1, w_1) \cdot (z_2, w_2) := z_1\bar{z}_2 + w_1\bar{w}_2.
$$

Identifying $\mathbb{C}^2 \cong \mathbb{R}^4$ it is easy to see that $\mathbb{S}^3 \subseteq \mathbb{R}^4$ corresponds to the unit sphere $S_1(0)$ of all $(z,w) \in \mathbb{C}^2$ at distance 1 to $0 \in \mathbb{C}^2$. Further, $\mathrm{SU}(2,\mathbb{C})$ acts on $\mathbb{C}^2$ isometrically by definition whence it also acts on $S_1(0)$. We claim that this action is free, transitive and smooth.

Smoothness is immediate. Note that every vector $v \in S_1(0)$ can be completed to a (positively oriented) orthonormal basis $\{v, w\}$ of $\mathbb{C}^2$. Then the matrix g with column vectors v and w is in $\mathrm{SU}(2,\mathbb{C})$ and $g.(1,0) = v$. Therefore the action is transitive. Finally, we will see that the action is free, i.e. it has trivial stabilizers. We will compute $\mathrm{Stab}_{\mathrm{SU}(2,\mathbb{C})}((1,0))$. Let

$$
g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Stab}((1,0)),
$$

i.e.

$$
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ -\bar{b} \end{pmatrix}
$$

whence $a = 1, b = 0$ and $g = I$. Therefore $\mathrm{Stab}((1,0)) = \{I\}$ and the action is free.

By exercise 3.1c) we get

$$
\mathrm{SU}(2,\mathbb{C}) \cong \mathrm{SU}(2,\mathbb{C})/\mathrm{Stab}((1,0)) \cong S_1(0) \cong \mathbb{S}^3.
$$

Because $\mathbb{S}^3$ is simply connected so is $\mathrm{SU}(2,\mathbb{C})$. In particular, $\mathrm{Ad} : \mathrm{SU}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{R})$ is the universal covering.

In order to see that $\mathrm{SO}(3,\mathbb{R}) \cong \mathbb{R}P^3$ recall that $\mathbb{R}P^3 \cong \mathbb{S}^3/(x \sim -x)$. The action of $\mathrm{SU}(2,\mathbb{C})$ on $S_1(0) \cong \mathbb{S}^3$ descends to a smooth action of $\mathrm{SU}(2,\mathbb{C})$ on $\mathbb{R}P^3 \cong S_1(0)/((z,w) \sim (-z,-w))$ by linearity. However, this action is no longer free. Indeed, the point stabilizer of $[1:0] \in S_1(0)/((z,w) \sim (-z,-w))$ is $\{\pm I\}$ and

$$\mathbb{R}P^3 \cong S_1(0)/((z,w) \sim (-z,-w)) \cong \mathrm{SU}(2,\mathbb{C})/\{\pm I\} \cong \mathrm{SO}(3,\mathbb{R}).$$

It is well known from topology that $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}/2\mathbb{Z}$ whence the fundamental group of $\mathrm{SO}(3,\mathbb{R})$ is also isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(d) Any Lie group G admits a simply connected Lie group $\tilde{G}$ covering it via some smooth covering homomorphism $\pi : \tilde{G} \to G$ as we have seen in Exercise 2 of Exercise sheet 2. This is the unique universal covering group of G with Lie algebra $\mathfrak{g}$ whence if $\mathfrak{g} \cong \mathfrak{su}(2,\mathbb{C})$ its universal covering group $\tilde{G}$ is isomorphic to $\mathrm{SU}(2,\mathbb{C})$. Therefore, G is isomorphic to $\mathrm{SU}(2,\mathbb{C})/\ker \pi$. We have seen in class that $N := \ker \pi$ is a discrete (normal) subgroup of $\mathrm{SU}(2,\mathbb{C})$. Since $\mathrm{SU}(2,\mathbb{C})$ is connected N has to be central. Hence, in order to see which Lie groups $G = \mathrm{SU}(2,\mathbb{C})/N$ have Lie algebra $\mathfrak{su}(2,\mathbb{C})$ it is enough to analyze the central discrete subgroups of $\mathrm{SU}(2,\mathbb{C})$.

To this end we analyze the centre

$$Z(G) = \{h \in G : hgh^{-1} = g \quad \forall g \in G\}$$

of $G = \mathrm{SU}(2,\mathbb{C})$. It is easy to see that $Z(G) = \ker \mathrm{Ad}$. As we have computed before $\ker \mathrm{Ad} = \{\pm I\}$. And its only subgroups are $\{I\}$ and $\{\pm I\}$. Therefore $\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{SU}(2,\mathbb{C})/\{\pm I\}$ and $\mathrm{SU}(2,\mathbb{C})$ are the *only* Lie groups with Lie algebra isomorphic to $\mathfrak{su}(2,\mathbb{C})$.