Functional Analysis I
Exercise Sheet 1

Exercise 1.1 Let X be a vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Prove the following statements.

(a) Every norm $\|\cdot\| : X \to [0, +\infty)$ on X induces a metric d on X by $d(x, y) := \|x - y\|$, for every $x, y \in X$.

(b) A metric $d : X \times X \to [0, +\infty)$ on X is induced by a norm (in the sense that there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$, for every $x, y \in X$) if and only if d is *homogeneous* and *translation invariant*, i.e.

$$d(x + v, y + v) = d(x, y) \qquad \forall\, x, y, v \in X,$$
$$d(\lambda x, \lambda y) = |\lambda| d(x, y) \qquad \forall\, x, y \in X,\ \forall \lambda \in \mathbb{K}.$$

(c) The operations of *scalar multiplication* $\cdot : \mathbb{K} \times X \to X$ and *addition* $+ : X \times X \to X$ are continuous with respect to the topology on X induced by any norm.

(d) The topologies on X induced by two equivalent norms coincide.

Exercise 1.2 Let $C^0([0, 1])$ be the set of the $\mathbb{R}$-valued continuous functions on $[0, 1]$. Prove the following statements.

(a) $(C^0([0, 1]), \|\cdot\|_\infty)$ is complete as a normed vector space over $\mathbb{R}$, where

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|, \qquad \forall\, f \in C^0([0, 1]).$$

(b) For every $p \in [1, +\infty)$, $(C^0([0, 1]), \|\cdot\|_p)$ is **not** complete as a normed vector space over $\mathbb{R}$, where

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad \forall\, f \in C^0([0, 1]).$$

(c) Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence. Given any $p \in [1, +\infty]$, let

$$\|(a_n)_{n\in\mathbb{N}}\|_{\ell^p} := \begin{cases} \left(\sum_{n=0}^{+\infty} |a_n|^p \right)^{\frac{1}{p}} & \text{if } p < +\infty \\ \sup_{n\in\mathbb{N}} |a_n| & \text{if } p = +\infty. \end{cases}$$

and define

$$\ell^p := \{(a_n)_{n\in\mathbb{N}} \text{ s.t. } \|(a_n)_{n\in\mathbb{N}}\|_{\ell^p} < +\infty\}.$$

Prove that $(\ell^p, \|\cdot\|_q)$ is a complete vector space over $\mathbb{R}$ if and only if $1 \leq q \leq p \leq +\infty$.

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Exercise 1.3 Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following are equivalent.

(a) $(X, \|\cdot\|)$ is a Banach space.

(b) For every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $\sum_{n=0}^{+\infty} \|x_n\| < +\infty$ the limit $\lim_{N\to+\infty} \sum_{n=0}^{N} x_n$ exists.

Exercise 1.4 Let $(X, \|\cdot\|)$ be an infinite dimensional normed vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Prove that there exists a non-continuous linear map $\ell : X \to \mathbb{K}$.