Exercise 1.1 Let X be a vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Prove the following statements.

(a) Every norm $\|\cdot\| : X \to [0, +\infty)$ on X induces a metric d on X by $d(x, y) := \|x - y\|$, for every $x, y \in X$.

(b) A metric $d : X \times X \to [0, +\infty)$ on X is induced by a norm (in the sense that there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$, for every $x, y \in X$) if and only if d is *homogeneous* and *translation invariant*, i.e.

$$\begin{aligned} d(x + v, y + v) &= d(x, y) \qquad \forall\, x, y, v \in X, \\ d(\lambda x, \lambda y) &= |\lambda| d(x, y) \qquad \forall\, x, y \in X, \forall \lambda \in \mathbb{K}. \end{aligned}$$

(c) The operations of *scalar multiplication* $\cdot : \mathbb{K} \times X \to X$ and *addition* $+ : X \times X \to X$ are continuous with respect to the topology on X induced by any norm.

(d) The topologies on X induced by two equivalent norms coincide.

Solution.

(a) Let $\|\cdot\|$ be a norm on X. We want to show that $d(x, y) := \|x - y\|$ for every $x, y \in X$ defines a metric on X.

Positivity. Clearly $d(x, y) \geq 0$. Moreover, $d(x, y) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

Symmetry. For every $x, y \in X$, we have

$$d(x, y) = \|x - y\| = \|-(y - x)\| = |-1| \|y - x\| = \|y - x\| = d(y, x).$$

Triangle inequality. For every $x, y, z \in X$, we have

$$d(x, y) = \|x - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y).$$

(b) Let d be a metric on X. We claim that that $\|\cdot\|_d := d(\,\cdot\,, 0) : X \times X \to [0, +\infty)$ is a norm on X such that $d(x, y) = \|x - y\|$, for every $x, y \in X$. First, we show that $\|\cdot\|_d$ is a norm. The positivity of $\|\cdot\|_d$ is clear.

Homogeneity. Let $\lambda \in \mathbb{K}$ and $x \in X$. Then

$$\|\lambda x\|_d = d(\lambda x, 0) = d(\lambda x, \lambda 0) = |\lambda| d(x, 0) = |\lambda| \|x\|_d.$$

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Triangle inequality. Let $x, y \in X$. Then

$$\|x+y\|_d = d(x+y,0) \le d(x+y,0+y) + d(y,0) = d(x,0) + d(y,0) = \|x\|_d + \|y\|_d.$$

Thus, we have proved that $\|\cdot\|_d$ is a norm on X. Moreover, for every $x, y \in X$, we have

$$\|x-y\|_d = d(x-y,0) = d(x+(-y), y+(-y)) = d(x,y).$$

Our claim follows. Since the fact that the distance d induced by a norm on X as in point (a) is translation invariant and homogeneous is straightforward, the full statement follows.

(c) Let $\|\cdot\|$ be any norm on X. Consider a sequence $((\lambda_n, x_n))_{n\in\mathbb{N}} \subset \mathbb{K} \times X$ such that $(\lambda_n, x_n) \to (\lambda, x) \in \mathbb{K} \times X$. Then

$$\begin{aligned}\|\lambda_n x_n - \lambda x\| &\le \|\lambda_n x_n - \lambda_n x\| + \|\lambda_n x - \lambda x\| \\ &= |\lambda_n|\|x_n - x\| + |\lambda_n - \lambda|\|x\| \to 0 \qquad (n \to +\infty).\end{aligned}$$

Now let $((x_n, y_n))_{n\in\mathbb{N}} \subset X \times X$ be such that $(x_n, y_n) \to (x, y) \in X \times X$. Then

$$\|(x_n + y_n) - (x+y)\| \le \|x_n - x\| + \|y_n - y\| \to 0 \qquad (n \to +\infty).$$

(d) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ e two equivalent norms on X inducing respectively the topologies τ_1 and τ_2 on X. Let $U \in \tau_1$. We want to show that $U \in \tau_2$. Indeed, fix any $x \in U$. Since $U \in \tau_1$, there exists $r_1 > 0$ such that $\{y \in X \text{ s.t. } \|y-x\|_1 < r_1\} \subset U$. Since $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$, there exists $C > 0$ such that $C\|z\|_2 \le \|z\|_1$, for every $z \in X$. Hence,

$$\left\{y \in X \text{ s.t. } \|y-x\|_2 < r_2 := \frac{r_1}{C}\right\} \subset \{y \in X \text{ s.t. } \|y-x\|_1 < r_1\} \subset U.$$

Thus, we have shown that for every given $x \in U$ there exists $r_2 > 0$ such that $\{y \in X \text{ s.t. } \|y-x\|_2 < r_2\} \subset U$. Our claim follows.

Analogously, we show that if $U \in \tau_2$ then $U \in \tau_1$. Hence $\tau_1 = \tau_2$.

□

Exercise 1.2 Let $C^0([0,1])$ be the set of the $\mathbb{R}$-valued continuous functions on $[0,1]$. Prove the following statements.

(a) $(C^0([0,1]), \|\cdot\|_\infty)$ is complete as a normed vector space over $\mathbb{R}$, where

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|, \qquad \forall\, f \in C^0([0,1]).$$

(b) For every $p \in [1, +\infty)$, $(C^0([0,1]), \|\cdot\|_p)$ is **not** complete as a normed vector space over $\mathbb{R}$, where

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad \forall\, f \in C^0([0,1]).$$

(c) Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence. Given any $p \in [1, +\infty]$, let

$$\|(a_n)_{n\in\mathbb{N}}\|_{\ell^p} := \begin{cases} \left(\displaystyle\sum_{n=0}^{+\infty} |a_n|^p \right)^{\frac{1}{p}} & \text{if } p < +\infty \\ \displaystyle\sup_{n\in\mathbb{N}} |a_n| & \text{if } p = +\infty. \end{cases}$$

and define

$$\ell^p := \{(a_n)_{n\in\mathbb{N}} \text{ s.t. } \|(a_n)_{n\in\mathbb{N}}\|_{\ell^p} < +\infty\}.$$

Prove that $(\ell^p, \|\cdot\|_q)$ is a complete vector space over $\mathbb{R}$ if and only if $1 \le q \le p \le +\infty$.

Solution.

(a) Let $(f_n)_{n\in\mathbb{N}} \subset C^0([0,1])$ be a Cauchy sequence with respect to $\|\cdot\|_\infty$. Then, for every $x \in [0,1]$ the sequence $(f_n(x))_{n\in\mathbb{N}} \subset \mathbb{R}$ is Cauchy. By completeness of $\mathbb{R}$, there exists $f(x) \in \mathbb{R}$ such that $f_n(x) \to f(x)$. We claim that $f_n \to f$ with respect to $\|\cdot\|_\infty$. Indeed, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$, for every $n, m \in \mathbb{N}$ such that $n, m \ge N$. Hence,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon, \qquad \forall\, n, m \ge N,\, \forall\, x \in [0,1].$$

By letting $m \to +\infty$ in the previous inequality we get

$$|f_n(x) - f(x)| < \varepsilon, \qquad \forall\, n \ge N,\, \forall\, x \in [0,1].$$

Finally, by taking the supremum over $x \in [0,1]$, we get

$$\|f_n - f\|_\infty < \varepsilon, \qquad \forall\, n \ge N.$$

Hence, our claim follows. Since f is the uniform limit of sequence of continuous functions, $f \in C^0([0,1])$. This concludes the proof of the statement.

(b) Fix any $p \in [1, +\infty)$. Consider the sequence $(f_n)_{n\in\mathbb{N}} \subset C^0([0,1])$ given by

$$f_n(x) := \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2} - \frac{1}{n}\right], \\ \left(\frac{n}{2}x + q_n\right)^{1/p} & \text{if } x \in \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right], \\ 1 & \text{if } x \in \left[\frac{1}{2} + \frac{1}{n}, 1\right], \end{cases}$$

where

$$q_n := \frac{n}{2}\left(\frac{1}{n} - \frac{1}{2}\right) = \frac{2-n}{4} \qquad \forall\, n \in \mathbb{N}.$$

Define also

$$f(x) := \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right), \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Notice that

$$\begin{aligned} \|f_n - f\|_p^p &= \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \left(\frac{n}{2}x + q_n\right) dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left(1 - \frac{n}{2}x - q_n\right) dx \\ &= \frac{n}{4}\left[x^2\right]_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} + \frac{q_n}{n} - \frac{n}{4}\left[x^2\right]_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} + \frac{1-q_n}{n} \\ &= \frac{1}{n} - \frac{n}{4}\frac{2}{n^2} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \to 0 \qquad (n \to +\infty). \end{aligned}$$

Hence, the sequence $(f_n)_{n\in\mathbb{N}} \subset C^0([0,1])$ converges to the discontinuous function f with respect to the L^p-norm. This is enough to show that $(C^0([0,1]), \|\cdot\|_p)$ is not a Banach space.

(c) First we show that $(\ell^p, \|\cdot\|_q)$ is a complete for $1 \le q \le p \le +\infty$. We proceed as follows.

Step 1. We claim that if $(a^{(k)})_{k\in\mathbb{N}} \subset \ell^p$ is a Cauchy sequence with respect to $\|\cdot\|_{\ell^q}$ then $a_n^{(k)} \to a_n$ as $k \to +\infty$ for every fixed $n \in \mathbb{N}$. Indeed, for every $n \in \mathbb{N}$ we have

$$|a_n^{(k)} - a_n^{(h)}| \le \|a^{(k)} - a^{(h)}\|_{\ell^q} \to 0 \qquad (k, h \to +\infty).$$

Hence, $(a_n^{(k)})_{k\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{R}$ for every $n \in \mathbb{N}$. By completeness of $\mathbb{R}$, $a_n^{(k)} \to a_n$ as $k \to +\infty$ for some $a_n \in \mathbb{R}$. Our claim follows

Step 2. Define the sequence $a := (a_n)_{n\in\mathbb{N}}$. We claim that $\|a^{(k)} - a\|_{\ell^q} \to 0$ as $k \to +\infty$. Indeed, given any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $\|a^{(k)} - a^{(h)}\|_{\ell^q} < \varepsilon$ whenever $k, h \geq K$. Thus, we have

$$\left(\sum_{n=0}^{N} |a_n^{(k)} - a_n^{(h)}|^q\right)^{\frac{1}{q}} \leq \|a^{(k)} - a^{(h)}\|_{\ell^q} < \varepsilon \qquad \forall\, k, h \geq K, \forall\, N \in \mathbb{N}.$$

in case $q < +\infty$ and

$$|a_n^{(k)} - a_n^{(h)}| \leq \|a^{(k)} - a^{(h)}\|_{\ell^q} < \varepsilon \qquad \forall\, k, h \geq K, \forall\, n \in \mathbb{N},$$

in case $q = +\infty$. By taking the limit as $h \to +\infty$ in the previous inequalities we get

$$\left(\sum_{n=0}^{N} |a_n^{(k)} - a_n|^q\right)^{\frac{1}{q}} \leq \|a^{(k)} - a^{(h)}\|_{\ell^q} < \varepsilon \qquad \forall\, k \geq K, \forall\, N \in \mathbb{N}.$$

in case $q < +\infty$ and

$$|a_n^{(k)} - a_n| \leq \|a^{(k)} - a^{(h)}\|_{\ell^q} < \varepsilon \qquad \forall\, k \geq K, \forall\, n \in \mathbb{N}.$$

By passing to the limit as $N \to +\infty$ in case $q < +\infty$ and taking the supremum over $n \in \mathbb{N}$ in case $q = +\infty$ we finally get

$$\|a^{(k)} - a\|_{\ell^q} < \varepsilon \qquad \forall\, k \geq K.$$

This concludes the proof of the claim.

Step 3. We claim that $\|x\|_{\ell^p} \leq 1$, for every $x \in \ell^q$ such that $\|x\|_{\ell^q} = 1$. If $q = p$ the statement is trivial so we just focus on the case $q < p$. If $p = +\infty$, then we notice that

$$|x_n| = \left(|x_n|^q\right)^{\frac{1}{q}} \leq \|x\|_{\ell^q} = 1,$$

which implies $\|x\|_{\ell^\infty} \leq 1$. If $p < +\infty$, then we notice that $|x_n|^p \leq |x_n|^q$ for every $n \in \mathbb{N}$ because $|x_n| \leq 1$ for every $n \in \mathbb{N}$ and $q < p$. Hence

$$\|x\|_{\ell^p} = \left(\sum_{n=0}^{+\infty} |x_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=0}^{+\infty} |x_n|^q\right)^{\frac{1}{p}} = \|x\|_{\ell^q}^{\frac{q}{p}} = 1.$$

Hence, our claim follows.

Step 4. We claim that $a \in \ell^p$. First we notice that $a \in \ell^q$. Indeed, fix any $k_0 \in \mathbb{N}$ such that $\|a^{(k)} - a\|_{\ell^q} \leq 1$ and notice that

$$\|a\|_{\ell^q} \leq \|a^{(k_0)}\|_{\ell^q} + \|a^{(k_0)} - a\|_{\ell^q} \leq \|a^{(k_0)}\|_{\ell^q} + 1 < +\infty.$$

But then, by defining

$$x = \frac{a}{\|a\|_{\ell^q}}$$

and applying Step 3 to $x \in \ell^q$ we get $\|x\|_{\ell^p} \leq 1$, which implies $\|a\|_{\ell^p} \leq \|a\|_{\ell^q} < +\infty$. The claim follows and we have proved the completeness of the normed vector space $(\ell^p, \|\cdot\|_q)$ whenever $1 \leq q \leq p \leq +\infty$.

Now, we want to show that $(\ell^p, \|\cdot\|_q)$ is **not** complete if $1 \leq p < q \leq +\infty$. First we face the case $q < +\infty$ and we consider the sequence $(a^{(k)})_{k\in\mathbb{N}} \subset \ell^p$ given by

$$a_n^{(k)} := \left(\frac{1}{n}\right)^{\frac{1}{p}+\frac{1}{k}}, \qquad \forall\, k \in \mathbb{N}, \forall\, n \in \mathbb{N}.$$

By dominated convergence one easily proves that $a^{(k)} \to a = (a_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{\ell^q}$, with

$$a_n := \left(\frac{1}{n}\right)^{\frac{1}{p}}, \qquad \forall\, n \in \mathbb{N}.$$

Since $a \notin \ell^p$, we conclude that $(\ell^p, \|\cdot\|_q)$ is not complete if $1 \leq p < q < +\infty$. In case $q = +\infty$, we consider the sequence $(a^{(k)})_{k\in\mathbb{N}} \subset \ell^p$ given by

$$a_n^{(k)} := \begin{cases} \left(\frac{1}{n}\right)^{\frac{1}{p}} & \text{if } n < k, \\ 0 & \text{if } n \geq k, \end{cases} \qquad \forall\, k \in \mathbb{N}, \forall\, n \in \mathbb{N}.$$

One easily proves that $a^{(k)} \to a = (a_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{\ell^\infty}$, with

$$a_n := \left(\frac{1}{n}\right)^{\frac{1}{p}}, \qquad \forall\, n \in \mathbb{N}.$$

Since $a \notin \ell^p$, we conclude that $(\ell^p, \|\cdot\|_\infty)$ is not complete if $1 \leq p < +\infty$.

□

Exercise 1.3 Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following are equivalent.

(a) $(X, \|\cdot\|)$ is a Banach space.

(b) For every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $\sum_{n=0}^{+\infty} \|x_n\| < +\infty$ the limit $\lim_{N\to+\infty} \sum_{n=0}^{N} x_n$ exists.

Solution. First we show that (a) implies (b). Consider a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\sum_{n=0}^{+\infty} \|x_n\| < +\infty$. Define

$$s_N := \sum_{n=0}^{N} x_n \qquad \forall\, N \in \mathbb{N}.$$

Notice that

$$\|s_N - s_M\| \leq \sum_{n=N}^{M} \|x_n\|, \qquad \forall\, N, M \in \mathbb{N} \text{ s.t. } M \geq N.$$

Hence, since $\sum_{n=0}^{+\infty} \|x_n\| < +\infty$, we conclude that the sequence $(s_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, $(s_N)_{N\in\mathbb{N}}$ has a limit in X and (b) follows.

Now we show that (b) implies (a). Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a Cauchy sequence in X. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, we can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset X$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| \leq \frac{1}{2^k} \qquad \forall\, k \in \mathbb{N}.$$

Consider the sequence $(v_k)_{k\in\mathbb{N}} \subset X$ given by

$$v_k := x_{n_{k+1}} - x_{n_k} \qquad \forall\, k \in \mathbb{N}.$$

Notice that

$$\sum_{k=0}^{+\infty} \|v_k\| = \sum_{k=0}^{+\infty} \|x_{n_{k+1}} - x_{n_k}\| \leq \sum_{k=0}^{+\infty} \frac{1}{2^k} = 2 < +\infty.$$

Hence, by (b) we conclude that the sequence $(s_k)_{k\in\mathbb{N}} \subset X$ given by

$$s_k := \sum_{j=0}^{k-1} v_j = x_{n_k} - x_{n_0}$$

has a limit $s \in X$. Thus,

$$\lim_{k\to+\infty} x_{n_k} = x_{n_0} + \lim_{k\to+\infty} (x_{n_k} - x_{n_0}) = x_{n_0} + \lim_{k\to+\infty} s_k = x_{n_0} + s =: x \in X.$$

We claim that $(x_n)_{n\in\mathbb{N}}$ converges to x. Indeed, by triangle inequality, we have

$$\|x_n - x\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\| \qquad \forall\, n, k \in \mathbb{N}.$$

Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, by letting first $n, k \to +\infty$ in the previous inequality we conclude that

$$\|x_n - x\| \to 0 \qquad (n \to +\infty)$$

and (a) follows. □

Exercise 1.4 Let $(X, \|\cdot\|)$ be an infinite dimensional normed vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Prove that there exists a non-continuous linear map $\ell : X \to \mathbb{K}$.

Solution. Let $\mathscr{B} = \{e_i\}_{i\in I}$ be a Hamel basis of X such that $\|e_i\| = 1$, for every $i \in I$. Since $\mathscr{B}$ is a infinite set by assumption) we can find a countable subset $\{e_{i_n}\}_{n\in\mathbb{N}} \subset \mathscr{B}$. We set $W := \operatorname{span}(\mathscr{B} \smallsetminus \{e_{i_n}\}_{n\in\mathbb{N}})$ and we define

$$\begin{aligned} \ell(e_{i_n}) &:= n \qquad \forall\, n \in \mathbb{N}, \\ \ell(w) &:= 0 \qquad \forall\, w \in W. \end{aligned}$$

The previous definitions give a unique linear map $\ell : X \to \mathbb{K}$. We claim that ℓ is discontinuous. Indeed, consider the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ given by

$$x_n := \frac{e_{i_n}}{n}, \qquad \forall\, n \in \mathbb{N}.$$

Clearly, $x_n \to 0$ in X. Nevertheless, $\ell(x_n) = 1$ for every $n \in \mathbb{N}$. This contradicts the continuity of of ℓ and our claim follows. □