**Exercise 10.1** Let $k \geq j$. Then the inclusion map $C^k([0,1]) \to C^j([0,1])$ is compact if and only if $k > j$.

**Exercise 10.2** Let $m \in \mathbb{N}$ and let $\emptyset \neq \Omega \subset \mathbb{R}^m$ be a bounded open set. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$, consider the linear operator $K : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x,y)f(y)dy.$$

(a) Prove that $K$ is well-defined, i.e., $Kf \in L^2(\Omega, \mathbb{C})$ for any $f \in L^2(\Omega, \mathbb{C})$.

(b) Prove that $K$ is a compact operator.

(c) If, in addition, the kernel $k$ satisfies $k(x,y) = \overline{k(y,x)}$ for almost every $(x,y) \in \Omega \times \Omega$, prove that the operator $A : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$, defined by

$$Af = f - Kf,$$

is surjective if and only if it is injective.

**Exercise 10.3** Let $X, Y$ be Banach spaces. Show that a bounded linear operator $A\colon X \to Y$ is Fredholm if and only if it is "invertible modulo compact operators", i.e. there exist $B_1, B_2 \in L(Y,X)$ and compact operators $K_1 \in L(Y)$, $K_2 \in L(X)$ so that

$$AB_1 = I - K_1, \quad B_2A = I - K_2.$$

**Exercise 10.4** Suppose that $X, Y, Z$ are Banach spaces, let $P \in L(X,Y)$ and assume that there exists a compact map $J \in L(X,Z)$. Suppose also that there is a constant $C > 0$ such that for all $x \in X$ one has

$$\|x\|_X \leq C\left(\|Px\|_Y + \|Jx\|_Z\right). \tag{1}$$

(a) If $P$ is injective, show that there is another constant $C' > 0$ such that for all $x \in X$ one has
$$\|x\|_X \leq C'\|Px\|_Y.$$

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(b) Without assuming that $P$ is injective show that (1) implies that $\ker(P)$ has finite dimension. Hence, prove the existence of a closed subspace $W$ of $X$ with $X = \ker(P) \oplus W$ (i.e. a topological complement $W$ of $\ker(P)$ in $X$). Then exploit part (a) to show that for all $x \in W$ one has

$$\|x\|_X \leq C'' \|Px\|_Y$$

for some constant $C'' > 0$.

(c) Assume $Z'$ is yet another Banach space, and there exist a compact operator $J' \in L(Y^*, Z')$ and a constant $C > 0$ so that for all $y^* \in Y^*$ we have

$$\|y^*\|_{Y^*} \leq C\Big(\|P^* y^*\|_{X^*} + \|J' y^*\|_{Z'}\Big).$$

Show that $P$ is Fredholm, and that also $P^*$ is Fredholm.