Exercise 10.1 Let $k \geq j$. Then the inclusion map $C^k([0,1]) \to C^j([0,1])$ is compact if and only if $k > j$.

Solution. First, notice that the inclusion $C^k([0,1]) \to C^k([0,1])$ is just the identity map on $C^k([0,1])$. Since space $C^k([0,1])$ is an infinite dimensional normed vector space, the identity map cannot be compact. Hence, we just need to show that the inclusion $C^k([0,1]) \to C^j([0,1])$ is compact if $k > j$ in order to get our statement.

Second, notice that the inclusion $C^k([0,1]) \to C^j([0,1])$ is continuous for every $k \geq j$, Indeed,

$$\|u\|_{C^j([0,1])} \leq \|u\|_{C^k([0,1])} \qquad \forall\, u \in C^k([0,1]) \tag{1}$$

whenever $k \geq j$. Hence, we just need to show that the inclusion $C^{k+1}([0,1]) \to C^k([0,1])$ is compact for every $k \in \mathbb{N}$ in order to get our statement. Indeed, once we have shown this, given any $k, \in \mathbb{N}$ with $k > j$ it just suffices to factorise $C^k([0,1]) \to C^{k-1}([0,1]) \to C^j([0,1])$. Since the first inclusion is compact and the second is continuous, we get that their composition is compact and we are done.

We reduced ourself to prove that that the inclusion $C^{k+1}([0,1]) \to C^k([0,1])$ is compact for every $k \in \mathbb{N}$. We argue by induction on $k \in \mathbb{N}$.

Base of the induction. We need to show that the inclusion $C^1([0,1]) \to C^0([0,1])$ is compact. This is a consequence of the Arzelà-Ascoli theorem which was already proved in class.

Induction step. Assume that the inclusion $C^{j+1}([0,1]) \to C^j([0,1])$ is compact for every $j = 0, ..., k$. We want to show that $C^{k+2}([0,1]) \to C^{k+1}([0,1])$ is compact. Pick any sequence $\{u_n\}_{n\in\mathbb{N}}$ lying in the unit ball in $C^{k+2}([0,1])$. By (1), we have that $\{u_n\}_{n\in\mathbb{N}}$ lies in the unit ball in $C^{k+1}([0,1])$. By the induction hypothesis, there exists a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ that converges to a limit $u \in C^k([0,1])$ with respect to the norm on $C^k([0,1])$. Now notice that the sequence $\{u_{n_i}^{(k+1)}\}_{i\in\mathbb{N}}$ lies in the unit ball in $C^1([0,1])$. Hence, there exists a subsequence (not relabeled) such that $\{u_{n_i}^{(k+1)}\}_{i\in\mathbb{N}}$ converges uniformly in $C^0([0,1])$ to a limit $v \in C^0([0,1])$. Now we fix any $x \in (0,1)$ and $h > 0$ such that $x + h \in [0,1]$ and we estimate

$$\begin{aligned}
|u^{(k)}(x+h) - u^{(k)}(x) - hv(x)| &= |u^{(k)}(x+h) - u_{n_i}^{(k)}(x+h)| + |u_{n_i}^{(k)}(x) - u^{(k)}(x)| \\
&\quad + |u_{n_i}^{(k)}(x+h) - u_{n_i}^{(k)}(x) - hv(x)| \\
&\leq 2\|u_{n_i}^{(k)} - u^{(k)}\|_{C^0([0,1])} + \left| \int_x^{x+h} u_{n^i}^{(k+1)}(t) - v(x)\, dt \right|
\end{aligned}$$

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$$
\begin{aligned}
&\leq 2\|u_{n_i}^{(k)} - u^{(k)}\|_{C^0([0,1])} + \int_x^{x+h} |u_{n^i}^{(k+1)}(t) - v(x)|\, dt \\
&\leq 2\|u_{n_i}^{(k)} - u^{(k)}\|_{C^0([0,1])} + \int_x^{x+h} |u_{n^i}^{(k+1)}(t) - v(t)|\, dt \\
&\quad + \int_x^{x+h} |v(t) - v(x)|\, dt \\
&\leq 2\|u_{n_i}^{(k)} - u^{(k)}\|_{C^0([0,1])} + \|u_{n_i}^{(k+1)} - v\|_{C^0([0,1])} h \\
&\quad + \int_x^{x+h} |v(t) - v(x)|\, dt.
\end{aligned}
$$

By letting $i \to +\infty$ in the previous estimate and then dividing both sides by h, we get

$$
\left| \frac{u^{(k)}(x+h) - u^{(k)}(x)}{h} - v(x) \right| \leq \frac{1}{h} \int_x^{x+h} |v(t) - v(x)|\, dt.
$$

By continuity of v, we get that

$$
\limsup_{h \to 0^+} \left| \frac{u^{(k)}(x+h) - u^{(k)}(x)}{h} - v(x) \right| \leq \limsup_{h \to 0^+} \frac{1}{h} \int_x^{x+h} |v(t) - v(x)|\, dt = 0.
$$

This implies that $u^{(k)}$ is differentiable at the point x and its derivative at x is $v(x)$, for every $x \in (0,1)$. Hence, we have $u^{(k+1)} = v \in C^0([0,1])$. We conclude both $u \in C^{k+1}([0,1])$ and $\{u_{n_i}\}_{i \in \mathbb{N}}$ converges to u in the norm on $C^{k+1}([0,1])$. The statement follows. □

Exercise 10.2 Let $m \in \mathbb{N}$ and let $\emptyset \neq \Omega \subset \mathbb{R}^m$ be a bounded open set. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$, consider the linear operator $K : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$
(Kf)(x) = \int_\Omega k(x,y) f(y) dy.
$$

(a) Prove that K is well-defined, i.e., $Kf \in L^2(\Omega, \mathbb{C})$ for any $f \in L^2(\Omega, \mathbb{C})$.

(b) Prove that K is a compact operator.

(c) If, in addition, the kernel k satisfies $k(x,y) = \overline{k(y,x)}$ for almost every $(x,y) \in \Omega \times \Omega$, prove that the operator $A : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$, defined by

$$
Af = f - Kf,
$$

is surjective if and only if it is injective.

Solution.

(a) Let $f \in L^2(\Omega, \mathbb{C})$. Then Hölder's inequality and Tonelli's theorem imply

$$\begin{aligned}\int_\Omega |(Kf)(x)|^2 dx &= \int_\Omega \left| \int_\Omega k(x,y) f(y) dy \right|^2 dx \leq \int_\Omega \left(\int_\Omega |k(x,y)f(y)| dy \right)^2 dx \\ &\leq \int_\Omega \left(\int_\Omega |k(x,y)|^2 dy \right) \|f\|^2_{L^2(\Omega)} dx = \|k\|^2_{L^2(\Omega\times\Omega)} \|f\|^2_{L^2(\Omega)}\end{aligned}$$

Since $k \in L^2(\Omega \times \Omega, \mathbb{C})$ by assumption, $\|Kf\|_{L^2(\Omega,\mathbb{C})} \leq \|k\|_{L^2(\Omega\times\Omega,\mathbb{C})} \|f\|_{L^2(\Omega,\mathbb{C})} < \infty$ follows.

(b) Being a Hilbert space, $L^2(\Omega, \mathbb{C})$ is reflexive. Part (d) of Exercise 9.4 implies that $K : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ is a compact operator if K maps weakly convergent sequences to norm-convergent sequences.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega, \mathbb{C})$ such that $f_n \xrightarrow{w} f$ as $n \to \infty$ for some $f \in L^2(\Omega, \mathbb{C})$. Since $k \in L^2(\Omega\times\Omega, \mathbb{C})$, Fubini's theorem implies that $k(x, \cdot) \in L^2(\Omega, \mathbb{C})$ for almost every $x \in \Omega$. Weak convergence therefore implies

$$(Kf_n)(x) = \langle k(x,\cdot), f_n \rangle_{L^2(\Omega,\mathbb{C})} \xrightarrow{n\to\infty} \langle k(x,\cdot), f \rangle_{L^2(\Omega,\mathbb{C})} = (Kf)(x)$$

for almost every $x \in \Omega$. As weakly convergent sequence, $(f_n)_{n\in\mathbb{N}}$ is bounded: there exists $C \in \mathbb{R}$ such that $\|f_n\|_{L^2(\Omega,\mathbb{C})} \leq C$ for every $n \in \mathbb{N}$. By Hölder's inequality,

$$|(Kf_n)(x)| \leq \int_\Omega |k(x,y) f_n(y)| \, dy \leq \|k(x,\cdot)\|_{L^2(\Omega)} \|f_n\|_{L^2(\Omega)} \leq C \|k(x,\cdot)\|_{L^2(\Omega)}.$$

The assumption $k \in L^2(\Omega \times \Omega, \mathbb{C})$ and Fubini's theorem imply that (the equivalence class of) the function $x \mapsto C\|k(x,\cdot)\|_{L^2(\Omega,\mathbb{C})}$ is in $L^2(\Omega, \mathbb{C})$. Thus, $(Kf_n)(x)$ is dominated by a function in $L^2(\Omega, \mathbb{C})$. Since $(Kf_n)(x)$ converges pointwise for almost every $x \in \Omega$ and since (Kf_n) is dominated by a function in $L^2(\Omega, \mathbb{C})$, Lebesgue's dominated convergence theorem implies L^2-convergence $\|Kf_n - Kf\|_{L^2(\Omega,\mathbb{C})} \to 0$ as $n \to \infty$.

(c) For $f, g \in L^2(\Omega, \mathbb{C})$ using repeatedly Fubini's theorem we compute:

$$
\begin{aligned}
(Kf,g)_{L^2} &= \int_\Omega Kf(x)\overline{g(x)}dx \\
&= \int_\Omega \left(\int_\Omega k(x,y)f(y)dy\right)\overline{g(x)}dx \\
&= \int_{\Omega\times\Omega} k(x,y)\overline{g(x)}f(y)dxdy \\
&= \int_\Omega f(y)\left(\int_\Omega k(x,y)\overline{g(x)}dx\right)dy \\
&= \int_\Omega f(y)\overline{\left(\int_\Omega \overline{k(x,y)}g(x)dx\right)}dy = (f,K^*g)_{L^2}\,,
\end{aligned}
$$

that is,

$$
\left(K^*g\right)(x) = \int_\Omega \overline{k(y,x)}g(y)dy
$$

Hence, under the additional assumption that $k(x,y)=\overline{k(y,x)}$ for a.a. $x,y\in\Omega$, the bounded operator K is self-adjoint. Therefore, the operator $A=(1-K):L^2(\Omega)\to L^2(\Omega)$ is also self-adjoint.

According to (b), K is a compact operator, which implies that the operator $A=(1-K)$ has closed image $\operatorname{im}(A)\subseteq H$. According to Banach's closed range theorem, this is equivalent to $\operatorname{im}(A)=\ker\left(A^*\right)^\perp$. Since $A^*=A$, we conclude in our setting that

$$
A \text{ surjective } \Leftrightarrow H=\operatorname{im}(A)=\ker(A)^\perp \Leftrightarrow \ker(A)=\{0\} \Leftrightarrow A \text{ injective.}
$$

□

Exercise 10.3 Let X,Y be Banach spaces. Show that a bounded linear operator $A\colon X\to Y$ is Fredholm if and only if it is "invertible modulo compact operators", i.e. there exist $B_1,B_2\in L(Y,X)$ and compact operators $K_1\in L(Y)$, $K_2\in L(X)$ so that

$$
AB_1 = I-K_1, \quad B_2A = I-K_2.
$$

Solution. First, we show that if A is invertible modulo compact operators then A is Fredholm. We need to show that $\dim\ker A<+\infty$ and $\dim\operatorname{coker} A<+\infty$. Notice that if $x\in\ker A$ then $x\in\ker(I-K_2)$. Hence, $\ker A$ is a subspace of $\ker(I-K_2)$. But since K_2 is compact, we have $\dim\ker(I-K_2)<+\infty$ and this implies $\dim\ker A<+\infty$.

On the other hand, if $x \in \operatorname{ran}(I - K_1)$ then $x \in \operatorname{ran}(A)$. Thus, $\operatorname{ran}(I - K_1)$ is a subspace of $\operatorname{ran} A$, which implies that $\operatorname{coker} A$ is a subspace of $\operatorname{coker}(I - K_1)$. But since K_1 is compact, we have $\dim \operatorname{coker}(I - K_2) < +\infty$ and this implies $\dim \operatorname{coker} A < +\infty$.

Now we turn to the other implication. Assume that A is Fredholm. Recall (see the solution of Exercise 3.1), the a closed vector subspace W of a Banach space is topologically complemented if and only if there exists a bounded linear operator P on such space such that $P^2 = P$ and $\operatorname{ran} P = W$. We call such an operator the *projection operator* on W.

Since $\ker A$ is finite-dimensional, $\ker A$ is topologically complemented in X (see Exercise 3.1-(a)). We call $P_1 \in L(X)$ the projection operator on $\ker A$ and we let $V := (I - P_1)(X)$ be the topological complement of $\ker A$ in X. Since $\operatorname{coker} A$ is finite-dimensional, $\operatorname{ran} A$ is topologically complemented in Y (see Exercise 3.1-(b)). We call $P_2 \in L(Y)$ the projection operator on $\operatorname{ran} A$. Now, notice that both V and $\operatorname{ran} A$ are closed since they are topologically complemented. Moreover, $A : V \to \operatorname{ran} A$ is linear, bijective and continuous. Hence, there is a continuous inverse $A^{-1} : \operatorname{ran} A \to V$. We define then $B_1, B_2 \in L(Y, X)$ by $B_1 := (I - P_1)A^{-1}$ and $B_2 := A^{-1}P_2$. We define $K_1 := P_1$ and $K_2 := I - P_2$. Notice that both K_1 and K_2 have finite rank, hence they are both compact. Moreover,

$$
\begin{aligned}
B_1 A &= I - P_1 = I - K_1, \\
A B_2 &= P_2 = I - (I - P_2) = I - K_2.
\end{aligned}
$$

The statement follows. □

Exercise 10.4 Suppose that X, Y, Z are Banach spaces, let $P \in L(X, Y)$ and assume that there exists a compact map $J \in L(X, Z)$. Suppose also that there is a constant $C > 0$ such that for all $x \in X$ one has

$$
\|x\|_X \leq C \left(\|Px\|_Y + \|Jx\|_Z \right). \tag{2}
$$

(a) If P is injective, show that there is another constant $C' > 0$ such that for all $x \in X$ one has

$$
\|x\|_X \leq C' \|Px\|_Y.
$$

(b) Without assuming that P is injective show that (2) implies that $\ker(P)$ has finite dimension. Hence, prove the existence of a closed subspace W of X with $X = \ker(P) \oplus W$ (i.e. a topological complement W of $\ker(P)$ in X). Then exploit part (a) to show that for all $x \in W$ one has

$$
\|x\|_X \leq C'' \|Px\|_Y
$$

for some constant $C'' > 0$.

(c) Assume Z' is yet another Banach space, and there exist a compact operator $J' \in L(Y^*, Z')$ and a constant $C > 0$ so that for all $y^* \in Y^*$ we have

$$\|y^*\|_{Y^*} \leq C\Big(\|P^*y^*\|_{X^*} + \|J'y^*\|_{Z'}\Big).$$

Show that P is Fredholm, and that also P^* is Fredholm.

Solution.

(a) For the sake of a contradiction, assume the claimed inequality is false: thus for any $k \geq 1$ one can find $x_k \in X$ with $\|x_k\|_X = 1$ and $\|Px_k\|_Y \leq \frac{1}{k}$. By compactness of the map $J : X \to Z$ one can find $\Lambda \subset \mathbb{N}$ such that

$$Jx_k \to z_\infty \text{ in } (Z, \|\cdot\|_Z) \quad (k \to \infty, k \in \Lambda)$$

At this stage, using (2) with $x_l - x_m$ in lieu of x, namely

$$\|x_l - x_m\|_X \leq C\left(\|P(x_l - x_m)\|_Y + \|J(x_l - x_m)\|_Z\right).$$

one gets that the sequence $(x_k)_{k\in\Lambda}$ is Cauchy in $(X, \|\cdot\|_X)$ so by completeness $x_k \to x_\infty$ in $(X, \|\cdot\|_X)$ $\quad (k \to \infty, k \in \Lambda)$. Since $P \in L(X, Y)$ we have

$$x_k \to x_\infty \Longrightarrow Px_k \to Px_\infty \quad (k \to \infty, k \in \Lambda)$$

but one the other hand $Px_k \to 0$ by construction, so we conclude $Px_\infty = 0$ and hence, by injectivity $x_\infty = 0$. However it should be $\|x_\infty\|_X = 1$ by the fact that $\|x_k\|_X = 1$ for any $k \geq 1$, contradiction.

(b) Let us prove that $\ker(P)$ has finite dimension by showing that $B_1(0; \ker(P))$ is relatively compact in $(X, \|\cdot\|_X)$. To this scope, pick $(x_k)_{k\in\mathbb{N}} \subset \ker(P)$ a sequence with $\|x_k\|_X < 1$ and let us prove it has a converging subsequence. Observe that inequality (2), when restricted to $x \in \ker(P)$ takes the form

$$\|x\|_X \leq C\|Jx\|_Z.$$

Hence (arguing as above) one first gets $Jx_k \to z_\infty(k \to \infty, k \in \Lambda)$, by compactness of J, and then, by the inequality above, $(x_k)_{k\in\Lambda}$ is Cauchy in $(X, \|\cdot\|_X)$ hence convergent to x_∞.

At this stage, the fact that $\ker(P)$ is topologically complemented in X follows by Exercise 3.1-(a), so let us write $X = \ker(P) \oplus W$ with $W \subset X$ closed (recall that a topologically complemented space and its complement are always closed).

Lastly, the restricted operator $P^\rho : W \to Y$ is linear, bounded and one can invoke the result of part (i). With W in lieu of X and P^ρ in lieu of P to conclude that $\|x\|_X \leq C''\|Px\|_Z$ uniformly for $x \in W \subset X$, which completes the proof.

(c) By point (b), both $\ker P$ and $\ker P^*$ are finite-dimensional. By Exercise 9.2-(a) we have that $\ker P^* = (\operatorname{ran} P)^\perp$ and $\operatorname{ran} P = {}^\perp(\ker P^*)$. In particular, $\operatorname{ran} P$ is a closed subspace of Y. Hence, by Exercise 9.1 we have that $\ker P^* = (\operatorname{ran} P)^\perp \cong (Y/\operatorname{ran} P)^*$ and we conclude that $(Y/\operatorname{ran} P)^*$ is finite-dimensional. But then $(Y/\operatorname{ran} P)^{**}$ is finite-dimensional, since it has the same dimension of $(Y/\operatorname{ran} P)^*$. But then since $Y/\operatorname{ran} P$ embeds isometrically onto $(Y/\operatorname{ran} P)^{**}$, we get that $Y/\operatorname{ran} P$ is finite-dimensional as well, i.e. $\operatorname{coker} P$ is finite-dimensional. This is sufficies to show that P is Fredholm.

By Exercise 10.3, there exist $Q_1, Q_1 \in L(Y, X)$ and compact operators $K_1 \in L(Y)$, $K_2 \in L(Y)$ such that

$$
\begin{aligned}
PQ_1 = I - K_1 \quad &\Leftrightarrow \quad Q_1^* P^* = I - K_1^*,\\
Q_2 P = I - K_2 \quad &\Leftrightarrow \quad P^* Q_2^* = I - K_2^*.
\end{aligned}
$$

By Schauder's theorem, both K_1^* and K_2^* are compact. Hence, P^* is invertible modulo compact operators, which again by Exercise 10.3 implies that P^* is Fredholm. The statement follows.

□