Exercise 11.1

(a) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces, let $T \in L(X,Y)$ be compact and let $J \in L(Y,Z)$ be injective. Prove that for every $\varepsilon \in (0, +\infty)$, there exists $C \in [0, +\infty)$ such that

$$\|Tx\|_Y \leq \varepsilon \|x\|_X + C\|JTx\|_Z \qquad \forall\, x \in X.$$

(b) Use part (a) to show: for all $\varepsilon > 0$ there exists C so that for all $u \in H^2(\mathbb{S}^1)$, we have

$$\|u\|_{H^1(\mathbb{S}^1)} \leq \varepsilon \|u\|_{H^2(\mathbb{S}^1)} + C\|u\|_{L^2(\mathbb{S}^1)}.$$

Exercise 11.2 Let H be a separable Hilbert space, and let $K \in L(H)$ be a compact self-adjoint operator (i.e. $K = K^*$). The goal of this exercise is to show that K has an eigenvector.

(a) Show that there exists a vector $v_0 \in H$, $\|v_0\| = 1$, so that $\|Kv_0\| = \|K\|_{L(H)}$.

(b) If $w \in H$, $w \perp v_0$, show that the derivative of $(-1, 1) \ni t \mapsto \|K(v_0 + tw)\|^2$ at $t = 0$ is equal to 0. Conclude that K^2 has an eigenvector with eigenvalue $\lambda = \|K\|^2_{L(H)}$.

(c) Deduce from (b) that K has an eigenvector with eigenvalue $\lambda_0 \in \{\|K\|_{L(H)}, -\|K\|_{L(H)}\}$.

Exercise 11.3 Let H be a Hilbert space and let $K \in L(H)$ be a compact operator. Prove the following statements. The goal of this exercise is to give a direct, hands-on proof (i.e. without using the general theory of Fredholm operators) that the index of $I - K$ is 0 when K is a compact operator on a Hilbert space.

(a) $\dim(\ker(I - K)) < \infty$.

(b) $\operatorname{im}(I - K)$ is closed.

(c) $\operatorname{im}(I - K) = (\ker(I - K^*))^{\perp}$.

(d) $\ker(I - K) = \{0\}$ if and only if $\operatorname{im}(I - K) = H$.

Hint: For "($\Rightarrow$)", assume that $\ker(I - K) = \{0\}$ and $\operatorname{im}(I - K) \neq H$. Show that this assumption leads to the following chain of proper inclusions: $H \supsetneq (I - K)(H) \supsetneq (I - K)^2(H) \supsetneq (I - K)^3(H) \supsetneq \ldots$ choose now $(x_k)_{k \in \mathbb{N}} \subseteq H$ such that $\|x_k\| = 1$, $x_k \in (I - K)^k(H), x_k \in \Big((I - K)^{k+1}(H)\Big)^{\perp}$ and show that $Kx_k - Kx_l$ has norm greater or equal than 1 whenever $k < l$ because $Kx_k - Kx_l$ can be written as the difference of x_k and an element of $(I - K)^{k+1}(H)$. For " ($\Leftarrow$), dualize.

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(e) $\dim(\ker(I-K)) = \dim(\ker(I-K^*))$.

Hint: Assume by contradiction that $\dim(\ker(I-K)) < \dim\left(\operatorname{im}(I-K)^\perp\right)$. Construct an injective compact map $A_0 : \ker(I-K) \to \operatorname{im}(I-K)^\perp$. Show that this map is not surjective. Extend A_0 to a compact map $A : H \to \operatorname{im}(I-K)^\perp$ with $\operatorname{im}(A) = \operatorname{im}(A_0)$ by setting $A|_{(\ker(I-K))^\perp} \equiv 0$. Show that $\ker(I-K-A) = \{0\}$, but $\operatorname{im}(I-K-A) \neq H$. This contradiction now shows $\dim(\ker(I-K)) \geq \dim\left(\operatorname{im}(I-K)^\perp\right)$. Finish by dualizing.

Exercise 11.4 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces, let $A \in L(X,Y)$ and $B \in L(Y,Z)$ be Fredholm. Show that $BA \in L(X,Z)$ is Fredholm.