Exercise 11.1

(a) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces, let $T \in L(X,Y)$ be compact and let $J \in L(Y,Z)$ be injective. Prove that for every $\varepsilon \in (0, +\infty)$, there exists $C \in [0, +\infty)$ such that

$$\|Tx\|_Y \leq \varepsilon \|x\|_X + C\|JTx\|_Z \qquad \forall\, x \in X.$$

(b) Use part (a) to show: for all $\varepsilon > 0$ there exists C so that for all $u \in H^2(\mathbb{S}^1)$, we have

$$\|u\|_{H^1(\mathbb{S}^1)} \leq \varepsilon \|u\|_{H^2(\mathbb{S}^1)} + C\|u\|_{L^2(\mathbb{S}^1)}.$$

Solution.

(a) Assume by contradiction that the claim is not true. Then there exist $\varepsilon \in (0, \infty)$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that

$$\|Tx_n\|_Y > \varepsilon \|x_n\|_X + n\,\|JTx_n\|_Z \text{ for all } n \in \mathbb{N}.$$

In particular, it holds for every $n \in \mathbb{N}$ that $Tx_n \neq 0$ so that the sequence $(x'_n)_{n\in\mathbb{N}} \subseteq X$, given by $x'_n = \frac{x_n}{\|Tx_n\|_Y}$ for all $n \in \mathbb{N}$, is well-defined and satisfies

$$1 = \|Tx'_n\|_Y > \varepsilon \|x'_n\|_X + n\,\|JTx'_n\|_Z \qquad \text{for all } n \in \mathbb{N}.$$

This implies that, on the one hand, $(x'_n)_{n\in\mathbb{N}} \subseteq X$ is bounded and, on the other hand, that $JTx'_n \to 0$ in Z as $n \to \infty$. The boundedness of $(x'_n)_{n\in\mathbb{N}} \subseteq X$ and the assumption that T is compact imply that there exists a subsequence $\left(x'_{n_k}\right)_{k\in\mathbb{N}}$ such that $\left(Tx'_{n_k}\right)_{k\in\mathbb{N}} \subseteq Y$ converges to some limit $y \in Y$ as $k \to \infty$. The fact that $\|Tx'_n\|_Y = 1$ for every $n \in \mathbb{N}$ implies that $\|y\|_Y = 1$. The assumed continuity of J, on the other hand, implies that $JTx'_{n_k} \to Jy$ in Z as $k \to \infty$. Since $JTx'_n \to 0$ as $n \to \infty$, we conclude that $Jy = 0$. By injectivity of J, we obtain $y = 0$. This, however, contradicts $\|y\|_Y = 1$, which we had already deduced above.

(b) It has been proven in class that the inclusion $I : H^2(\mathbb{S}^1) \to H^1(\mathbb{S}^1)$ is compact. Moreover, it is straightforward to see that the inclusion $J : H^1(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is injective. By part (a), the statement follows.

□

Exercise 11.2 Let H be a separable Hilbert space, and let $K \in L(H)$ be a compact self-adjoint operator (i.e. $K = K^*$). The goal of this exercise is to show that K has an eigenvector.

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(a) Show that there exists a vector $v_0 \in H$, $\|v_0\| = 1$, so that $\|Kv_0\| = \|K\|_{L(H)}$.

(b) If $w \in H$, $w \perp v_0$, show that the derivative of $(-1,1) \ni t \mapsto \|K(v_0 + tw)\|^2$ at $t = 0$ is equal to 0. Conclude that K^2 has an eigenvector with eigenvalue $\lambda = \|K\|^2_{L(H)}$.

(c) Deduce from (b) that K has an eigenvector with eigenvalue $\lambda_0 \in \{\|K\|_{L(H)}, -\|K\|_{L(H)}\}$.

Solution.

(a) Recall that

$$\|K\|_{L(H)} = \sup_{\|x\|_H = 1} \|Kx\|_H.$$

Let $(x_k)_{k\in\mathbb{N}}$ be such that $\|x_k\|_H = 1$ and $\|Kx_k\|_H \to \|K\|_{L(H)}$. Since H is reflexive we have that the unit ball in H is weakly sequentially compact. In particular, there exists a subsequence of $(x_k)_{k\in\mathbb{N}}$ (not relabeled) such that $x_k \overset{w}{\rightharpoonup} x$ in H. Since K is compact on a reflexive space, by Exercise we have that $Kx_k \to Kx$ strongly in H as $k \to +\infty$. Hence $\|Kx\|_H = \|K\|_{L(H)}$. Moreover, since the norm is weakly lower semicontinuous we have $\|x\|_H \leq 1$. We set

$$v_0 := \frac{x}{\|x\|_H}$$

and we notice that $\|v_0\|_H = 1$. Moreover,

$$\|Kv_0\|_H = \frac{\|Kx\|_H}{\|x\|_H} \geq \|K\|_{L(H)}.$$

By definition of $\|K\|_{L(H)}$, this implies $\|Kv_0\|_H = \|K\|_{L(H)}$.

(b) By point (a) we have

$$\frac{\|K(v_0 + tw)\|^2_H}{\|v_0 + tw\|^2_H} \leq \|K\|^2_{L(H)} = \|Kv_0\|^2_H, \qquad \forall\, t \in (-1,1).$$

Hence, the function

$$(-1,1) \ni t \mapsto \frac{\|K(v_0 + tw)\|^2_H}{\|v_0 + tw\|^2_H}$$

has a global maximum at $t = 0$. Hence, we have

$$\left.\frac{d}{dt}\right|_{t=0} \frac{\|K(v_0 + tw)\|^2_H}{\|v_0 + tw\|^2_H} = 0.$$

Since $w \perp v_0$, we have $\|v_0 + tw\|_H^2 = 1 + t^2\|w\|_H^2$, for every $t \in (-1, 1)$. Thus,

$$0 = \frac{d}{dt}\bigg|_{t=0} \frac{\|K(v_0 + tw)\|_H^2}{\|v_0 + tw\|_H^2} = \frac{d}{dt}\bigg|_{t=0} \|K(v_0 + tw)\|_H^2 = (Kv_0, Kw) = (K^2 v_0, w).$$

By arbitrariness of $w \in v_0^\perp$ we conclude that $K^2 v_0 = \alpha v_0$, for some $\alpha \in \mathbb{R}$. By scalar product with v_0 in the previous equality we get

$$\alpha = \alpha(v_0, v_0)_H = (K^2 v_0, v_0)_H = (Kv_0, Kv_0)_H = \|Kv_0\|_H^2 = \|K\|_{L(H)}^2.$$

The statement of point (b) follows.

(c) Let $\lambda := \|K\|_{L(H)}^2 \geq 0$.

$$0 = (K^2 - \lambda I)v_0 = (K - \sqrt{\lambda} I)(K + \sqrt{\lambda} I)v_0$$

Now if $w := (K + \sqrt{\lambda} I)v_0 \neq 0$, then w is an eigenvector of K with eigenvalue $\sqrt{\lambda} = \|K\|_{L(H)}$. If instead $(K + \sqrt{\lambda} I)v_0 = 0$, then v_0 is an eigenvector of K with eigenvalue $-\sqrt{\lambda} = -\|K\|_{L(H)}$. The statement follows.

$\square$

Exercise 11.3 Let H be a Hilbert space and let $K \in L(H)$ be a compact operator. Prove the following statements. The goal of this exercise is to give a direct, hands-on proof (i.e. without using the general theory of Fredholm operators) that the index of $I - K$ is 0 when K is a compact operator on a Hilbert space.

(a) $\dim(\ker(I - K)) < \infty$.

(b) $\mathrm{im}(I - K)$ is closed.

(c) $\mathrm{im}(I - K) = (\ker(I - K^*))^\perp$.

(d) $\ker(I - K) = \{0\}$ if and only if $\mathrm{im}(I - K) = H$.

Hint: For "($\Rightarrow$)", assume that $\ker(I - K) = \{0\}$ and $\mathrm{im}(I - K) \neq H$. Show that this assumption leads to the following chain of proper inclusions: $H \supsetneq (I - K)(H) \supsetneq (I - K)^2(H) \supsetneq (I - K)^3(H) \supsetneq \ldots$ choose now $(x_k)_{k \in \mathbb{N}} \subseteq H$ such that $\|x_k\| = 1$, $x_k \in (I - K)^k(H), x_k \in \left((I - K)^{k+1}(H)\right)^\perp$ and show that $Kx_k - Kx_l$ has norm greater or equal than 1 whenever $k < l$ because $Kx_k - Kx_l$ can be written as the difference of x_k and an element of $(I - K)^{k+1}(H)$. For " ($\Leftarrow$), dualize.

(e) $\dim(\ker(I-K)) = \dim(\ker(I-K^*))$.

Hint: Assume by contradiction that $\dim(\ker(I-K)) < \dim\left(\operatorname{im}(I-K)^\perp\right)$. Construct an injective compact map $A_0 : \ker(I-K) \to \operatorname{im}(I-K)^\perp$. Show that this map is not surjective. Extend A_0 to a compact map $A : H \to \operatorname{im}(I-K)^\perp$ with $\operatorname{im}(A) = \operatorname{im}(A_0)$ by setting $A|_{(\ker(I-K))^\perp} \equiv 0$. Show that $\ker(I-K-A) = \{0\}$, but $\operatorname{im}(I-K-A) \neq H$. This contradiction now shows $\dim(\ker(I-K)) \geq \dim\left(\operatorname{im}(I-K)^\perp\right)$. Finish by dualizing.

Solution.

(a) Assume that $\dim(\ker(I-K)) = \infty$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \ker(I-K)$ with $(x_n, x_m) = \delta_{nm}$ for all $n, m \in \mathbb{N}$. In particular, $(x_n)_{n\in\mathbb{N}}$ does not have a converging subsequence. By compactness of K and by $x_n = Kx_n$ for every $n \in \mathbb{N}$, the sequence $(x_n)_{n\in\mathbb{N}}$ should have a converging subsequence, though.

Alternatively, restricting K to the closed (and therefore complete) subspace $\ker(I-K)$, we are in the situation of a Hilbert/Banach space on which the identity operator is a compact operator or, put differently, in which the closed unit ball is compact. This only ever happens in finite dimensions.

(b) We claim that there exists $\gamma \in (0,\infty)$ so that $\|x\| \leq \gamma\|x - Kx\|$ for all $x \in (\ker(I-K))^\perp$. Indeed, if this was not the case, then there would exist a sequence $(x_n)_{n\in\mathbb{N}} \subseteq (\ker(I-K))^\perp$ satisfying $1 = \|x_n\| > n\,\|x_n - Kx_n\|$ for all $n \in \mathbb{N}$. This would imply that $x_n - Kx_n \to 0$ as $n \to \infty$. On the other hand, by compactness of K, we may assume (by passing to a subsequence, if necessary) that $Kx_n \to y$ as $n \to \infty$ for some $y \in H$. Consequentially, we would have that $x_n = (x_n - Kx_n) + Kx_n \to 0 + y = y$ as $n \to \infty$. Hence, we would obtain $y \in (\ker(I-K))^\perp$, $\|y\| = \lim_{n\to\infty}\|x_n\| = 1$, and $Ky = \lim_{n\to\infty} Kx_n = y$. But this is not possible as $y \in (\ker(I-K))^\perp$ and $Ky = y$ (i.e., $y \in \ker(I-K)$) would imply that $y = 0$, contradicting $\|y\| = 1$.

With $\gamma \in (0,\infty)$ so that $\|x\| \leq \gamma\|x - Kx\|$ for all $x \in (\ker(I-K))^\perp$, we can now conclude that $\operatorname{im}(I-K)$ is closed: Let $(y_n)_{n\in\mathbb{N}} \subseteq \operatorname{im}(I-K)$ be an arbitrary sequence converging to y_∞ in H. Let $(x_n)_{n\in\mathbb{N}} \subseteq H$ satisfy for all $n \in \mathbb{N}$ that $y_n = x_n - Kx_n$. Denoting by $P \in L(H)$ the orthogonal projection onto the closed subspace $(\ker(I-K))^\perp$, we obtain that $(Px_n)_{n\in\mathbb{N}} \subseteq (\ker(I-K))^\perp$ (and therefore $x_n - Px_n \in \ker(I-K)$) so that $Px_n - KPx_n = x_n - Kx_n = y_n$ for every $n \in \mathbb{N}$. Now, we can use the previously obtained inequality to verify that $(Px_n)_{n\in\mathbb{N}} \subseteq H$ is a Cauchy sequence:

$$\limsup_{N\to\infty}\sup_{m,n\geq N}\|Px_n - Px_m\| \leq \limsup_{N\to\infty}\sup_{m,n\geq N}\gamma\,\|y_n - y_m\| = 0.$$

Thus, there exists a limit $x_\infty \in H$ of $(Px_n)_{n\in\mathbb{N}}$ and $x_\infty - Kx_\infty = \lim_{n\to\infty}(I-K)Px_n = \lim_{n\to\infty} y_n = y_\infty$, i.e., $y_\infty \in \operatorname{im}(I-K)$.

(c) This follows immediately from the fact that $\overline{\operatorname{im}(I-K)} = (\ker(I-K^*))^\perp$ and the fact that $\operatorname{im}(I-K)$ is closed (cp. part (b)).

(d) "($\Rightarrow$)": Assume by contradiction that $\ker(I-K)=\{0\}$ and $\operatorname{im}(I-K)\neq H$. We first show by induction that $(I-K)^{k+1}(H) \subsetneq (I-K)^k(H)$ for every $k\in\mathbb{N}_0$. Indeed, for $k=0$, this is just the previous assumption. And if $k\in\mathbb{N}$ is such that $(I-K)^k(H)\subsetneq(I-K)^{k-1}(H)$ but $(I-K)^{k+1}(H)=(I-K)^k(H)$, then we obtain that $x_0\in(I-K)^{k-1}(H)\backslash(I-K)^k(H)$ gets mapped by $I-K$ to $(I-K)x_0\in(I-K)^k(H)=(I-K)^{k+1}(H)=(I-K)\Big((I-K)^k(H)\Big)$ so that there has to exist $x_1\in(I-K)^k(H)$ satisfying $(I-K)x_0=(I-K)x_1$. Hence, $0\neq x_0-x_1\in\ker(I-K)$ (since $x_0\neq x_1$ as $x_0\notin(I-K)^k(H)$ while $x_1\in(I-K)^k(H)$), which contradicts that $I-K$ is injective.

Knowing that - under the assumption that $\ker(I-K)=\{0\}$ and $\operatorname{im}(I-K)\neq H$ - it has to hold for every $k\in\mathbb{N}_0$ that $(I-K)^{k+1}(H)\subsetneq(I-K)^k(H)$ and since $(I-K)^k(H)$ is closed for every $k\in\mathbb{N}$ by part (b), we can now choose a sequence $(x_k)_{k\in\mathbb{N}}\subseteq H$ such that $\|x_k\|=1$ and $x_k\in(I-K)^k(H)\cap\Big((I-K)^{k+1}(H)\Big)^\perp$ for every $k\in\mathbb{N}$. Moreover, note that for all $k,l\in\mathbb{N}$ with $k<l$ it holds that

$$x_k-(Kx_k-Kx_l)=\underbrace{(x_k-Kx_k)}_{\in(I-K)^{k+1}(H)}-\underbrace{(x_l-Kx_l)}_{\in(I-K)^{l+1}(H)}+\underbrace{x_l}_{\in(I-K)^l(H)}\in(I-K)^{k+1}(H),$$

i.e., $\|Kx_k-Kx_l\|\geq\operatorname{dist}\Big(x_k,(I-K)^{k+1}(H)\Big)=\|x_k\|=1$ (since, sloppily speaking, Kx_k-Kx_l has to cover at least the part of x_k perpendicular to $(I-K)^{k+1}(H)\Big)$. In particular, $(Kx_k)_{k\in\mathbb{N}}$ does not have a converging subsequence, although $(x_k)_{k\in\mathbb{N}}\subseteq H$ is a bounded sequence and K is compact.

"($\Leftarrow$)": $\operatorname{im}(I-K)=H$ implies that $\ker(I-K^*)=\{0\}$. By Schauder's theorem K^* is compact. The previous part of the proof hence implies that $\operatorname{im}(I-K^*)=H$. Hence, $\ker(I-K)=\{0\}$.

(e) Assume for a contradiction that $\dim(\ker(I-K))<\dim(\ker(I-K^*))$. Since $\ker(I-K^*)=\operatorname{im}(I-K)^\perp$, we are assuming that $\dim(\ker(I-K))<\dim\Big(\operatorname{im}(I-K)^\perp\Big)$. Since $\ker(I-K)$ is finite-dimensional by part (a) and $\dim(\ker(I-K))<\dim(\operatorname{im}(I-K)^\perp$), there exists an injective, but not surjective map $A_0:\ker(I-K)\to\operatorname{im}(I-K)^\perp$. Moreover, since $\ker(I-K)$ is finite-dimensional, A_0 has finite rank and is therefore compact. Define $A:H\to\operatorname{im}(I-K)^\perp$ via $A(x+y)=A_0x$ for $x\in\ker(I-K)$, $y\in(\ker(I-K))^\perp$. Since A is a compact linear map, $K+A$ is also a linear map (from H to H). Note that $(I-K-A)x=0$ implies that $Ax=(I-K)x\in$

$\operatorname{im}(I-K) \cap (\operatorname{im}(I-K))^{\perp} = \{0\}$, hence $x \in \ker(I-K) \cap \ker(A) = \ker(A_0) = \{0\}$. On the other hand, for every $x \in H$ it holds that $(I-K-A)x = (I-K)x - Ax \in \operatorname{im}(I-K) \oplus \operatorname{im}(A) \subsetneq \operatorname{im}(I-K) \oplus (\operatorname{im}(I-K))^{\perp} = H$ since $\operatorname{im}(A) \subsetneq (\operatorname{im}(I-K))^{\perp}$. Hence, we have $\ker(I-K-A) = \{0\}$ and $\operatorname{im}(I-K-A) \neq H$, contradicting part (d). This contradiction now shows $\dim(\ker(I-K)) \geq \dim(\ker(I-K^*))$. Since K^* is, by Schauder's theorem, compact as well, we obtain by the above argument that $\dim(\ker(I-K^*)) \geq \dim(\ker(I-K))$.

□

Exercise 11.4 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces, let $A \in L(X,Y)$ and $B \in L(Y,Z)$ be Fredholm. Show that $BA \in L(X,Z)$ is Fredholm.

Solution. We use Exercise 10.3. Since A is Fredholm, and we find $A_1, A_2 \in L(Y,X)$ and compact operators $K_1 \in L(Y)$, $K_2 \in L(X)$ so that

$$AA_1 = I - K_1, \quad A_2A = I - K_2.$$

Again, since B is Fredholm we find $B_1, B_2 \in L(Z,Y)$ and compact operators $\tilde{K}_1 \in L(Z)$, $\tilde{K}_2 \in L(Y)$ so that

$$BB_1 = I - \tilde{K}_1, \quad B_2B = I - \tilde{K}_2.$$

We notice that

$$(BA)(A_1B_1) = B(AA_1)B_1 = B(I-K_1)B_1 = (B - BK_1)B_1 = I - (\tilde{K}_1 + BK_1B_1)$$
$$(A_2B_2)(BA) = A_2(B_2B)A = A_1(I-\tilde{K}_2)A = I - (K_2 + A_1\tilde{K}_2A).$$

Since both $\tilde{K}_1 + BK_1B_1$ and $K_2 + A_1\tilde{K}_2A$ are compact, by Exercise 10.3 the statement follows. □