Functional Analysis I
Exercise Sheet 12

Exercise 12.1 Let $H \subset L^2(\mathbb{S}^1)$ be given by $H = \operatorname{ran} P$, where $P : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the projection operator given by

$$(Pf)(\theta) := \sum_{n=0}^{+\infty} \hat{f}(n) e^{in\theta}, \qquad \forall\, \theta \in [0, 2\pi].$$

Given any $\varphi \in C^0(\mathbb{S}^1)$, we define the *Toeplitz operator* $T_\varphi : H \to H$ by $T_\varphi(u) := P(\varphi u)$, for every $u \in H$.

(a) Clearly, $\|T_\varphi\|_{L(H)} \leq \|\varphi\|_{L^\infty}$. By explicit computation, for $\varphi(\theta) = E_k(\theta) = e^{ik\theta}$ ($k \in \mathbb{N}$) show that $T_{E_k} T_{E_l} - T_{E_k E_l}$ is a compact operator on H for every $k, l \in \mathbb{Z}$.

(b) For every $\varphi, \psi \in C^0(\mathbb{S}^1)$, show that $T_\varphi T_\psi - T_{\varphi\psi}$ is a compact operator on H.

Hint. Approximate φ and ψ with linear combinations of exponentials.

(c) Prove that if $\varphi \in C^0(\mathbb{S}^1)$ is nowhere vanishing then T_φ is a Fredholm operator.

Hint. Show that a Fredholm inverse is given by T_ψ, with $\psi(\theta) = (\varphi(\theta))^{-1}$ for every $\theta \in [0, 2\pi]$.

(d) *Bonus problem.* A nowhere vanishing $\varphi \in C^0(\mathbb{S}^1)$ is said to have *degree* $k \in \mathbb{Z}$ if φ is homotopic to E_k through continuous maps of $\mathbb{S}^1$ to $\mathbb{C} \setminus \{0\}$. Show that this implies

$$\operatorname{index}(T_\varphi) = \operatorname{index}(T_{E_k}).$$

Compute this index by explicitly describing $\ker(T_{E_k})$ and $\ker(T^*_{E_k})$.

Exercise 12.2 Let X be a complex Banach space, and let $\Omega \subset \mathbb{C}$ be a non-empty open subset.

(a) Prove that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence if and only if $\{\lambda(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly for $\lambda \in X^*$ with $\|\lambda\| \leq 1$. That is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, we have $|\lambda(x_n) - \lambda(x_m)| < \varepsilon$ for $n, m \geq n_0$ *and for all* $\lambda \in X^*$, $\|\lambda\| \leq 1$.

(b) Suppose $f : \Omega \to X$ is *weakly holomorphic*, meaning that for all $\lambda \in X^*$ the complex-valued function $\lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic. Prove that f is holomorphic.

Last modified: 9 December 2022

Hint. Let $z_0 \in \Omega$. Write $\lambda(f(z_0))$ as an integral of $\lambda(f(z))/(z - z_0)$ over a small circle γ around z_0, and use this to show that

$$\lambda\left(\frac{f(z_0+h) - f(z_0)}{h}\right) - \frac{d}{dz}\bigg|_{z=z_0}(\lambda \circ f) = \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{h}\left(\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0}\right) - \frac{1}{(z-z_0)^2}\right]\lambda(f(z))\,dz.$$

Using Banach–Steinhaus, show that $f(z)$ is uniformly bounded for $z \in \gamma$, and use this and simple estimates for the above integral to show that $\lambda((f(z_0+h) - f(z_0))/h)$ is uniformly Cauchy for $\lambda \in X^*$, $\|\lambda\| \leq 1$.

(c) By using part (b) and similar arguments, prove the following result. Suppose $f\colon \Omega \to L(X)$ is weakly holomorphic in the sense that for all $x \in X$ and $\lambda \in X^*$ the function $\Omega \ni z \mapsto \lambda(f(z)x) \in \mathbb{C}$ is holomorphic. Show that $\Omega \ni z \mapsto f(z) \in L(X)$ is holomorphic.

Exercise 12.3 Let H be a complex Hilbert space, and let $A \in L(H)$ be a *normal operator*, that is, $AA^* = A^*A$. Show (using induction) that $\|A^n\| = \|A\|^n$ for all $n \in \mathbb{N}$. Deduce that the spectral radius of A is equal to $\|A\|_{L(H)}$.

Exercise 12.4 Let $k \in C^0([0,1] \times [0,1])$, and define the Volterra integral operator $A\colon C^0([0,1]) \to C^0([0,1])$ by

$$(Au)(x) = \int_0^x k(x,y)u(y)\,dy \qquad \forall\, x \in [0,1].$$

Compute the spectral radius of A.