Exercise 12.1 Let $H \subset L^2(\mathbb{S}^1)$ be given by $H = \operatorname{ran} P$, where $P : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the projection operator given by

$$(Pf)(\theta) := \sum_{n=0}^{+\infty} \hat{f}(n)e^{in\theta}, \qquad \forall\, \theta \in [0, 2\pi].$$

Given any $\varphi \in C^0(\mathbb{S}^1)$, we define the *Toeplitz operator* $T_\varphi : H \to H$ by $T_\varphi(u) := P(\varphi u)$, for every $u \in H$.

(a) Clearly, $\|T_\varphi\|_{L(H)} \leq \|\varphi\|_{L^\infty}$. By explicit computation, for $\varphi(\theta) = E_k(\theta) = e^{ik\theta}$ $(k \in \mathbb{N})$ show that $T_{E_k}T_{E_l} - T_{E_kE_l}$ is a compact operator on H for every $k, l \in \mathbb{Z}$.

(b) For every $\varphi, \psi \in C^0(\mathbb{S}^1)$, show that $T_\varphi T_\psi - T_{\varphi\psi}$ is a compact operator on H.

Hint. Approximate φ and ψ with linear combinations of exponentials.

(c) Prove that if $\varphi \in C^0(\mathbb{S}^1)$ is nowhere vanishing then T_φ is a Fredholm operator.

Hint. Show that a Fredholm inverse is given by T_ψ, with $\psi(\theta) = (\varphi(\theta))^{-1}$ for every $\theta \in [0, 2\pi]$.

(d) *Bonus problem.* A nowhere vanishing $\varphi \in C^0(\mathbb{S}^1)$ is said to have *degree* $k \in \mathbb{Z}$ if φ is homotopic to E_k through continuous maps of $\mathbb{S}^1$ to $\mathbb{C} \setminus \{0\}$. Show that this implies

$$\operatorname{index}(T_\varphi) = \operatorname{index}(T_{E_k}).$$

Compute this index by explicitly describing $\ker(T_{E_k})$ and $\ker(T^*_{E_k})$.

Solution.

(a) By direct computation we get

$$((T_{E_k}T_{E_l})(u))(\theta) = \sum_{n=k}^{+\infty} \hat{u}(n-(k+l))e^{in\theta}, \qquad \forall\, \theta \in [0, 2\pi],$$
$$((T_{E_kE_l})(u))(\theta) = \sum_{n=0}^{+\infty} \hat{u}(n-(k+l))e^{in\theta}, \qquad \forall\, \theta \in [0, 2\pi].$$

Hence

$$((T_{E_k}T_{E_l} - T_{E_kE_l})(u))(\theta) = \begin{cases} -\displaystyle\sum_{n=0}^{k-1} \hat{u}(n-(k+l))e^{in\theta} & \text{if } k > 0 \\ 0 & \text{if } k \leq 0. \end{cases}$$

Thus, $T_{E_k}T_{E_l} - T_{E_kE_l}$ is compact because it has finite rank.

Last modified: 10 January 2023

(b) By density, we can find $\{\varphi_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ of the form

$$\varphi_j(\theta) = \sum_{k=1}^{N_{1,j}} a_k e^{in_k\theta}, \qquad \forall\,\theta \in [0,2\pi],$$
$$\psi_j(\theta) = \sum_{k=1}^{N_{2,j}} b_k e^{im_k\theta}, \qquad \forall\,\theta \in [0,2\pi],$$

and such that $\varphi_j \to \varphi$, $\psi_j \to \psi$ uniformly on $[0,2\pi]$ as $j \to +\infty$. Notice that

$$T_{\varphi_j}T_{\psi_j} = \sum_{k=1}^{N_{1,j}}\sum_{h=1}^{N_{2,j}} a_k b_h T_{E_{n_k}} T_{E_{m_h}},$$
$$T_{\varphi_j\psi_j} = \sum_{k=1}^{N_{1,j}}\sum_{h=1}^{N_{2,j}} a_k b_h T_{E_{n_k}E_{m_h}},$$

for every $j \in \mathbb{N}$. Hence, by point (a) we have that

$$T_{\varphi_j}T_{\psi_j} - T_{\varphi_j\psi_j} = \sum_{k=1}^{N_{1,j}}\sum_{h=1}^{N_{2,j}} a_k b_h (T_{E_{n_k}} T_{E_{m_h}} - T_{E_{n_k}E_{m_h}}),$$

is compact for every $i \in \mathbb{N}$ because it is a finite linear combination of compact operators. Moreover,

$$\begin{aligned}
\|(T_{\varphi_j}T_{\psi_j} - T_{\varphi_j\psi_j}) - (T_\varphi T_\psi - T_{\varphi\psi})\|_{L(H)} &\le \|(T_{\varphi_j} - T_\varphi)T_{\psi_j}\|_{L(H)} + \|T_\varphi(T_{\psi_j} - T_\psi)\|_{L(H)} \\
&\quad + \|T_{\varphi_j\psi_j} - T_{\varphi\psi}\|_{L(H)} \\
&= \|(T_{\varphi_j-\varphi})T_{\psi_i}\|_{L(H)} + \|T_\varphi(T_{\psi_j-\psi})\|_{L(H)} \\
&\quad + \|T_{\varphi_j\psi_j-\varphi\psi}\|_{L(H)} \\
&\le \|\varphi_j - \varphi\|_{L^\infty}\|\psi_j\|_{L^\infty} + \|\varphi\|_{L^\infty}\|\psi_j - \psi\|_{L^\infty} \\
&\quad + \|\varphi_j\psi_j - \varphi\psi\|_{L^\infty} \to 0 \quad (j \to +\infty).
\end{aligned}$$

Thus, $T_\varphi T_\psi - T_{\varphi\psi}$ is compact because it is the limit of sequence of compact operators.

(c) By point (b), we have that

$$T_\varphi T_\psi - \mathrm{Id}_H = T_\varphi T_\psi - T_{\varphi\psi} =: K_1,$$
$$T_\psi T_\varphi - \mathrm{Id}_H = T_\psi T_\varphi - T_{\psi\varphi} =: K_2$$

are compact operators on H. Hence, by Exercise 10.3, We have that T_φ is compact because it is invertible modulo compact operators.

(d) By hypothesis, there exists a continuous homotopy $H : \mathbb{S}^1 \times [0,1] \to \mathbb{C} \smallsetminus \{0\}$ such that such that $H(0,\theta) = \varphi(\theta)$ and $H(1,\theta) = E_k(\theta)$, for every $\theta \in [0, 2\pi]$. Consider the path $\gamma : [0,1] \to \mathrm{Fred}(H)$ given by $\gamma_t := T_{H(t,\cdot)}$. Since

$$C^0(\mathbb{S}^1, \mathbb{C} \smallsetminus \{0\}) \ni \varphi \mapsto T_\varphi \in \mathrm{Fred}(H)$$

is a continuous linear map (see (a) e.g.) and

$$[0,1] \ni t \mapsto H(t,\cdot) \in C^0(\mathbb{S}^1, \mathbb{C} \smallsetminus \{0\})$$

is continuous by definition of homotopy, then we conclude that γ is a continuous path joining T_φ and T_{E_k} in $\mathrm{Fred}(H)$. As the Fredholm index is constant on the connected components of $\mathrm{Fred}(H)$ (see the lecture notes on polybox), we get that

$$\mathrm{index}(T_\varphi) = \mathrm{index}(T_{E_k}).$$

Now we want to compute $\mathrm{index}(T_{E_k})$ explicitly. In order to do this, we see that

$$0 \equiv ((T_{E_k})(u))(\theta) = \sum_{n=0}^{+\infty} \hat{u}(n-k)e^{in\theta}$$

if and only if $\hat{u}(n) = 0$ for every $n \in \mathbb{N}$ such that $n \geq -k$. Thus, since $u \in H$, we get

$$\dim\ker(T_{E_k}) = \begin{cases} -k & \text{if } k < 0, \\ 0 & \text{if } k \geq 0. \end{cases}$$

Notice that $T_{E_k}^* = T_{E_{-k}}$, for every $k \in \mathbb{Z}$. Hence,

$$\dim\ker(T_{E_k}^*) = \dim\ker(T_{E_{-k}}) = \begin{cases} k & \text{if } k > 0, \\ 0 & \text{if } k \leq 0. \end{cases}$$

In particular, we obtain that

$$\mathrm{index}(T_{E_k}) = \dim\ker(T_{E_k}) - \dim\ker(T_{E_k}^*) = -k, \qquad \forall\, k \in \mathbb{Z}.$$

□

Exercise 12.2 Let X be a complex Banach space, and let $\Omega \subset \mathbb{C}$ be a non-empty open subset.

(a) Prove that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence if and only if $\{\lambda(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly for $\lambda \in X^*$ with $\|\lambda\| \leq 1$. That is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, we have $|\lambda(x_n) - \lambda(x_m)| < \varepsilon$ for $n, m \geq n_0$ *and for all* $\lambda \in X^*$, $\|\lambda\| \leq 1$.

(b) Suppose $f : \Omega \to X$ is *weakly holomorphic*, meaning that for all $\lambda \in X^*$ the complex-valued function $\lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic. Prove that f is holomorphic.

Hint. Let $z_0 \in \Omega$. Write $\lambda(f(z_0))$ as an integral of $\lambda(f(z))/(z - z_0)$ over a small circle γ around z_0, and use this to show that

$$\lambda\left(\frac{f(z_0 + h) - f(z_0)}{h}\right) - \frac{d}{dz}\bigg|_{z=z_0} (\lambda \circ f) = \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{h}\Big(\frac{1}{z - (z_0 + h)} - \frac{1}{z - z_0}\Big) - \frac{1}{(z - z_0)^2}\right] \lambda(f(z))\, dz.$$

Using Banach–Steinhaus, show that $f(z)$ is uniformly bounded for $z \in \gamma$, and use this and simple estimates for the above integral to show that $\lambda((f(z_0+h) - f(z_0))/h)$ is uniformly Cauchy for $\lambda \in X^*$, $\|\lambda\| \le 1$.

(c) By using part (b) and similar arguments, prove the following result. Suppose $f\colon \Omega \to L(X)$ is weakly holomorphic in the sense that for all $x \in X$ and $\lambda \in X^*$ the function $\Omega \ni z \mapsto \lambda(f(z)x) \in \mathbb{C}$ is holomorphic. Show that $\Omega \ni z \mapsto f(z) \in L(X)$ is holomorphic.

Solution.

(a) First we show that if $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence then $\{\lambda(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly for $\lambda \in X^*$ with $\|\lambda\| \le 1$. Indeed, since $\{x_n\}_{n\in\mathbb{N}} \subset X$ is Cauchy, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\|_X < \varepsilon$ for every $n, m \in \mathbb{N}$ with $n, m \ge n_0$. Then, for every $\lambda \in X^*$ such that $\|\lambda\| \le 1$ and every $n, m \ge n_0$ we have

$$|\lambda(x_n) - \lambda(x_m)| = |\lambda(x_n - x_m)| \le \|\lambda\|\|x_n - x_m\|_X \le \|x_n - x_m\|_X < \varepsilon.$$

Now assume that $\{\lambda(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly for $\lambda \in X^*$ with $\|\lambda\| \le 1$. Then, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\substack{\lambda\in X^*\\ \|\lambda\|\le 1}} |\lambda(x_n) - \lambda(x_m)| < \varepsilon$$

for every $n, m \in \mathbb{N}$ with $n, m \ge n_0$. Thus,

$$\|x_n - x_m\|_X = \sup_{\substack{\lambda\in X^*\\ \|\lambda\|\le 1}} |\lambda(x_n - x_m)| = \sup_{\substack{\lambda\in X^*\\ \|\lambda\|\le 1}} |\lambda(x_n) - \lambda(x_m)| < \varepsilon,$$

for every $n, m \in \mathbb{N}$ with $n, m \ge n_0$. The statement follows.

(b) Fix any $z_0 \in \Omega$ and let γ be a circle of radius r centered at z_0 such that $\gamma \subset \Omega$. Let $\lambda \in X^*$. Notice that, since $\lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic, by the Cauchy integral formula we have

$$\lambda(f(z_0)) = \frac{1}{2\pi i}\int_\gamma \frac{\lambda(f(z))}{(z - z_0)}\, dz,$$

$$\frac{d}{dz}\bigg|_{z=z_0}(\lambda\circ f)=\frac{1}{2\pi i}\int_\gamma\frac{\lambda(f(z))}{(z-z_0)^2}\,dz,$$

and

$$\lambda(f(z_0+h))=\frac{1}{2\pi i}\int_\gamma\frac{\lambda(f(z))}{(z-(z_0+h))}\,dz,$$

for every $h\in\Omega$ with $|h|\leq\frac{r}{2}$. By linearity of gamma, we get

$$\begin{aligned}&\frac{\lambda(f(z_0+h))-\lambda(f(z_0))}{h}-\frac{d}{dz}\bigg|_{z=z_0}(\lambda\circ f)\\&=\frac{1}{2\pi i}\int_\gamma\left[\frac{1}{h}\left(\frac{1}{z-(z_0+h)}-\frac{1}{z-z_0}\right)-\frac{1}{(z-z_0)^2}\right]\lambda(f(z))\,dz\\&=\frac{1}{2\pi i}\int_\gamma\left(\frac{h}{(z-(z_0+h))(z-z_0)^2}\right)\lambda(f(z))\,dz.\end{aligned}\tag{1}$$

Let $J_X : X\to X^{**}$ be the standard linear isometry mapping $x\in X$ into the linear functional on X^* given by $\big(J_X(x)\big)(\lambda)=\lambda(x)$, for every $x\in X$ and $\lambda\in X^*$. Consider the family of linear functionals $\{J_X(f(z))\}_{z\in\gamma}$. Fix any $\lambda\in X^*$. Since $\lambda\circ f$ is holomorphic and hence continuous on the compact set γ, for every $\lambda\in X^*$ there exists C_λ such that $|\big(J_X(f(z))\big)(\lambda)|=|\lambda(f(z))|\leq C_\lambda$ for every $z\in\gamma$. This means that the family $\{J_X(f(z))\}_{z\in\gamma}\subset X^{**}$ is pointwise bounded on X^{**}. By the Banach-Steinhaus theorem, we have that $\{J_X(f(z))\}_{z\in\gamma}$ is uniformly bounded in X^{**}, i.e. there exists $C>0$ such that

$$\|f(z)\|_X=\|J_X(f(z))\|_{X^{**}}\leq C\qquad\forall\, z\in\gamma,$$

where the first equality comes from the fact that J_X is an isometry. This immediately implies that for every $\lambda\in X^*$ with $\|\lambda\|\leq 1$ we have

$$|\lambda(f(z))|\leq\|f(z)\|_X\leq C\qquad\forall\, z\in\gamma.$$

Hence, by (1), for every $h_1,h_2\in\mathbb{C}$ with $|h_1|,|h_2|\leq\frac{r}{2}$ we get

$$\left|\frac{\lambda(f(z_0+h_1))-\lambda(f(z_0))}{h_1}-\frac{\lambda(f(z_0+h_2))-\lambda(f(z_0))}{h_2}\right|\leq\frac{C}{\pi}\int_\gamma\frac{2}{r^3}\,dz=\frac{4C}{r^2}|h_1-h_2|.$$

In particular, we have that

$$\left\{\frac{\lambda(f(z_0+h))-\lambda(f(z_0))}{h}\right\}_{0<h\leq\frac{r}{2}}$$

is uniformly Cauchy for $\lambda \in X^*$ with $\|\lambda\| \le 1$. By point (a), this implies that

$$\left\{ \frac{f(z_0+h)-f(z_0)}{h} \right\}_{0<h\le \frac{r}{2}}$$

is Cauchy in X. This amounts to saying that f is differentiable at z_0. The statement follows.

(c) Fix any $z_0 \in \Omega$ and let γ be a circle of radius r centered at z_0 such that $\gamma \subset \Omega$. Let $\lambda \in X^*$ and $x \in X$. Notice that, since $\lambda \circ (f(\,\cdot\,)x) : \Omega \to \mathbb{C}$ is holomorphic, by the Cauchy integral formula we have

$$\lambda(f(z_0)x) = \frac{1}{2\pi i} \int_\gamma \frac{\lambda(f(z)x)}{(z-z_0)}\, dz,$$

$$\left.\frac{d}{dz}\right|_{z=z_0} (\lambda \circ (f(\,\cdot\,)x)) = \frac{1}{2\pi i} \int_\gamma \frac{\lambda(f(z)x)}{(z-z_0)^2}\, dz,$$

and

$$\lambda(f(z_0+h)x) = \frac{1}{2\pi i} \int_\gamma \frac{\lambda(f(z)x)}{(z-(z_0+h))}\, dz,$$

for every $h \in \Omega$ with $|h| \le \frac{r}{2}$. By linearity of gamma, we get

$$\begin{aligned}
&\frac{\lambda(f(z_0+h)x) - \lambda(f(z_0)x)}{h} - \left.\frac{d}{dz}\right|_{z=z_0} (\lambda \circ (f(\,\cdot\,)x)) \\
&= \frac{1}{2\pi i} \int_\gamma \left[\frac{1}{h}\left(\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} \right] \lambda(f(z)x)\, dz \\
&= \frac{1}{2\pi i} \int_\gamma \left(\frac{h}{(z-(z_0+h))(z-z_0)^2} \right) \lambda(f(z)x)\, dz. \qquad (2)
\end{aligned}$$

By part (b), we know that for every $x \in X$ the function $\Omega \ni z \mapsto f(z)x \in X$ is holomorphic. This implies that the family of linear operators $\{f(z)\}_{z\in\gamma}$ is pointwise bounded. By the Banach-Steinhaus theorem, we conclude that $\{f(z)\}_{z\in\gamma}$ is uniformly bounded, i.e. that there exists a constant $C > 0$ such that

$$\|f(x)\|_{L(X)} \le C.$$

This implies that for every $\lambda \in X^*$ and $x \in X$ such that $\|\lambda\|_{X^*} \le 1$ and $\|x\|_X \le 1$ we have

$$|\lambda(f(z)x)| \le \|\lambda\|_{X^*} \|f(z)\|_{L(X)} \|x\|_X \le C \qquad \forall\, z \in \gamma.$$

Hence, by (2), for every $h_1, h_2 \in \mathbb{C}$ with $|h_1|, |h_2| \leq \frac{r}{2}$ we get

$$\left| \frac{\lambda(f(z_0+h_1)x) - \lambda(f(z_0)x)}{h_1} - \frac{\lambda(f(z_0+h_2)x) - \lambda(f(z_0)x)}{h_2} \right| \leq \frac{C}{\pi} \int_\gamma \frac{2}{r^3}\, dz = \frac{4C}{r^2} |h_1 - h_2|.$$

In particular, we have that

$$\left\{ \frac{\lambda(f(z_0+h)x) - \lambda(f(z_0)x)}{h} \right\}_{0<h\leq \frac{r}{2}} = \left\{ \frac{\lambda((f(z_0+h) - f(z_0))x)}{h} \right\}_{0<h\leq \frac{r}{2}}$$

is uniformly Cauchy for $\lambda \in X^*$ and $x \in X$ with $\|\lambda\|_{X^*} \leq 1$ and $\|x\|_X \leq 1$. This easily implies that

$$\left\{ \frac{f(z_0+h) - f(z_0)}{h} \right\}_{0<h\leq \frac{r}{2}}$$

is Cauchy in X, by the same arguments that are used in point (a). The statement follows.

□

Exercise 12.3 Let H be a complex Hilbert space, and let $A \in L(H)$ be a *normal operator*, that is, $AA^* = A^*A$. Show (using induction) that $\|A^n\| = \|A\|^n$ for all $n \in \mathbb{N}$. Deduce that the spectral radius of A is equal to $\|A\|_{L(H)}$.

Solution. First we claim that $\|A^n\|_{L(H)} = \|A\|^n_{L(H)}$, for every $n \in \mathbb{N} \setminus \{0\}$. We proceed by induction on n. In case $n = 1$, the the is nothing to show. Now assume that $\|A^k\|_{L(H)} = \|A\|^k_{L(H)}$ for every $k = 1, ..., n \in \mathbb{N}$. Recall that, for every normal operator, it holds that $\|Tx\|_H = \|T^*x\|_H$ for every $x \in H$ (see Bemerkung 6.7.1. in Struwe's lecture notes). Hence, for every $x \in H$ with $\|x\|_H = 1$ we have

$$\begin{aligned} \|T^n x\|_H^2 = (T^n x, T^n x)_H = (T^* T^n x, T^{n-1} x)_H &\leq \|T^* T^n x\|_H \|T^{n-1} x\|_H \\ &\leq \|T^{n+1} x\|_H \|T^{n-1} x\|_H \leq \|T^{n+1}\|_{L(H)} \|T^{n-1}\|_{L(H)}. \end{aligned}$$

It follows that

$$\|T\|^{2n}_{L(H)} = \|T^n\|^2_{L(H)} \leq \|T^{n+1}\|_H \|T^{n-1}\|_H = \|T^{n+1}\|_{L(H)} \|T\|^{n-1}_{L(H)}.$$

Thus,

$$\|T\|^{n+1}_{L(H)} \leq \|T^{n+1}\|_{L(H)}.$$

Since the converse inequality always holds, we have that $\|T\|_{L(H)}^{n+1} = \|T^{n+1}\|_{L(H)}$ and our claim follows.

By Satz 6.5.3 in Struwe's lecture notes, we finally get

$$r_T := \lim_{n\to+\infty} \|T^n\|_{L(H)}^{\frac{1}{n}} = \|T\|_{L(H)}$$

and we are done. □

Exercise 12.4 Let $k \in C^0([0,1]\times[0,1])$, and define the Volterra integral operator $A\colon C^0([0,1]) \to C^0([0,1])$ by

$$(Au)(x) = \int_0^x k(x,y)u(y)\,dy \qquad \forall\, x \in [0,1].$$

Compute the spectral radius of A.

Solution. By direct computation, we have

$$(A^n u)(x) = \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} k(x,y_1)k(y_1,y_2)\dots k(y_{n-1},y_n)u(y_n)\,dy_n\dots\,dy_1, \quad \forall\, x \in [0,1].$$

Hence,

$$\|A^n u\|_{C^0} = \sup_{x\in[0,1]} \left| \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} k(x,y_1)k(y_1,y_2)\dots k(y_{n-1},y_n)u(y_n)\,dy_n\dots\,dy_1 \right|.$$

and it is trivial to see that

$$\begin{aligned}\|A^n\|_{L(C^0([0,1]))} &\le \sup_{x\in[0,1]} \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} |k(x,y_1)k(y_1,y_2)\dots k(y_{n-1},y_n)|\,dy_n\dots\,dy_1 \\ &\le \|k\|^n_{C^0([0,1]\times[0,1])} \sup_{x\in[0,1]} \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} dy_n\dots\,dy_1 = \frac{1}{n!}\|k\|^n_{C^0([0,1]\times[0,1])}\end{aligned}$$

Thus, by Stirling's formula, we get

$$r_A = \lim_{n\to+\infty} \|A^n\|_{L(C^0([0,1]))}^{\frac{1}{n}} \le \|k\|_{C^0([0,1]\times[0,1])} \lim_{n\to+\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0.$$

□