Exercise 2.1 Let

$$c_0 := \Big\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty \text{ s.t. } \lim_{n\to+\infty} a_n = 0\Big\} \subset \ell^\infty.$$

(a) Prove that $(c_0)^* \cong \ell^1$, i.e. show that there exists a surjective isometry $I : \ell^1 \to (c_0)^*$.

(b) Prove that $(\ell^1)^* \cong \ell^\infty$, i.e. show that there exists a surjective isometry $\tilde{I} : \ell^\infty \to (\ell^1)^*$.

(c) Prove that there exists a continuous and linear functional $\lambda : \ell^\infty \to \mathbb{R}$ such that

$$\liminf_{n\to+\infty} a_n \le \lambda((a_n)_{n\in\mathbb{N}}) \le \limsup_{n\to+\infty} a_n, \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Show that such functional is **not** of the form

$$\lambda((a_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{+\infty} x_n a_n, \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in \ell^\infty$$

for some sequence $(x_n)_{n\in\mathbb{N}} \in \ell^1$.

Exercise 2.2 Recall that a topological space X is called *separable* if it admits a countable and dense subset $S \subset X$.

(a) Let X be a Banach space. Show that if X^* is separable then X is separable.

(b) Prove that ℓ^∞ is not a separable Banach space.

(c) Prove that ℓ^1 is a separable Banach space.

(d) Prove that $(\ell^\infty)^* \ncong \ell^1$, i.e. show that there is no surjective isometry $I : \ell^1 \to (\ell^\infty)^*$.

Exercise 2.3 Show that the subspaces

$$U := \{(a_n)_{n\in\mathbb{N}} \in \ell^1 \text{ s.t. } a_{2n} = 0,\, \forall\, n \in \mathbb{N}\}$$
$$V := \{(a_n)_{n\in\mathbb{N}} \in \ell^1 \text{ s.t. } a_{2n-1} = n a_{2n},\, \forall\, n \in \mathbb{N} \setminus \{0\}\}$$

are both closed in ℓ^1 but $U \oplus V$ is **not** closed in ℓ^1.

Exercise 2.4 Let X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Prove that a linear functional $\lambda : X \to \mathbb{K}$ is continuous if and only if $\ker(\lambda)$ is a closed vector subspace of X.

(b) Prove that if $V \subset X$ is a closed vector subspace of X and $W \supset V$ is a vector subspace of X such that W/V is finite dimensional, then W is closed.

Last modified: 29 September 2022