**Exercise 2.1** Let

$$c_0 := \left\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty \text{ s.t. } \lim_{n\to+\infty} a_n = 0\right\} \subset \ell^\infty.$$

(a) Prove that $(c_0)^* \cong \ell^1$, i.e. show that there exists a surjective isometry $I : \ell^1 \to (c_0)^*$.

(b) Prove that $(\ell^1)^* \cong \ell^\infty$, i.e. show that there exists a surjective isometry $\tilde{I} : \ell^\infty \to (\ell^1)^*$.

(c) Prove that there exists a continuous and linear functional $\lambda : \ell^\infty \to \mathbb{R}$ such that

$$\liminf_{n\to+\infty} a_n \le \lambda((a_n)_{n\in\mathbb{N}}) \le \limsup_{n\to+\infty} a_n, \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Show that such functional is **not** of the form

$$\lambda((a_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{+\infty} x_n a_n, \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in \ell^\infty$$

for some sequence $(x_n)_{n\in\mathbb{N}} \in \ell^1$.

**Solution.**

(a) Given any sequence $a = (a_n)_{n\in\mathbb{N}} \in \ell^1$, we let $\lambda_a : c_0 \to \mathbb{R}$ be given by

$$\lambda_a(x) := \sum_{n=0}^{+\infty} a_n x_n, \qquad \forall\, x = (x_n)_{n\in\mathbb{N}} \in c_0.$$

Clearly, $\lambda_a$ is $\mathbb{R}$-linear. We notice that

$$|\lambda_a(x)| \le \sum_{n=0}^{+\infty} |a_n||x_n| \le \|a\|_{\ell^1}\|x\|_{\ell^\infty} < +\infty, \qquad \forall\, x = (x_n)_{n\in\mathbb{N}} \in c_0.$$

This immediately implies that $\lambda_a \in (c_0)^*$ is well-defined and $\|\lambda_a\|_{(c_0)^*} \le \|a\|_{\ell^1}$. Consider the sequence $(x^{(k)})_{k\in\mathbb{N}} \subset c_0$ given by

$$x_n^{(k)} := \begin{cases} \operatorname{sgn}(a_n) & \text{if } n \le k, \\ 0 & \text{if } n > k, \end{cases} \tag{1}$$

for every $n, k \in \mathbb{N}$. Notice that $\|x^{(k)}\|_{\ell^\infty} = 1$ for every $k \in \mathbb{N}$ and

$$\lambda_a(x^{(k)}) = \sum_{n=0}^{k} |a_n| \to \|a\|_{\ell^1} \qquad (k \to +\infty).$$

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Hence, we conclude $\|\lambda_a\|_{(c_0)^*} = \|a\|_{\ell^1}$.

We define the linear map $I : \ell^1 \to (c_0)^*$ given by $I(a) := \lambda_a$, for every $a \in \ell^1$. From what we have proved so far, it follows that $I$ is a linear isometry. In order to conclude, we just need to show that $I$ is surjective. Indeed, pick any $\lambda \in (c_0)^*$ and define $a = (a_n)_{n\in\mathbb{N}}$ by $a_n := \lambda(e^{(n)})$ with

$$e_k^{(n)} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases}$$

for every $n, k \in \mathbb{N}$. Notice that, for every given $k \in \mathbb{N}$, we have

$$\sum_{n=0}^{k} |a_n| = \lambda(x^{(k)}) \leq \|\lambda\|_{(c_0)^*},$$

where $(x^{(k)})_{k\in\mathbb{N}} \subset c_0$ is defined as in (1). By letting $k \to +\infty$ in the previous inequality we get

$$\|a\|_{\ell^1} \leq \|\lambda\|_{(c_0)^*} < +\infty,$$

i.e. $a \in \ell^1$. In order to conclude, we claim that $I(a) = \lambda_a = \lambda$. Indeed, first notice that for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$ we have

$$\left\| \sum_{n=0}^{k} x_n e^{(n)} - x \right\|_{\ell^\infty} = \sup_{n\geq k} |x_n| \to 0 \qquad (k \to +\infty).$$

Hence, by continuity and linearity of $\lambda$ we get

$$\begin{aligned} \lambda(x) &= \lambda\left( \lim_{k\to+\infty} \sum_{n=0}^{k} x_n e^{(n)} \right) = \lim_{k\to+\infty} \lambda\left( \sum_{n=0}^{k} x_n e^{(n)} \right) \\ &= \lim_{k\to+\infty} \sum_{n=0}^{k} x_n \lambda(e^{(n)}) = \lim_{k\to+\infty} \sum_{n=0}^{k} a_n x_n = \lambda_a(x), \qquad \forall\, x = (x_n)_{n\in\mathbb{N}} \in c_0. \end{aligned}$$

The statement follows.

(b) Given any sequence $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty$, we let $\lambda_a : \ell^1 \to \mathbb{R}$ be given by

$$\lambda_a(x) := \sum_{n=0}^{+\infty} a_n x_n, \qquad \forall\, x = (x_n)_{n\in\mathbb{N}} \in \ell^1.$$

Clearly, $\lambda_a$ is $\mathbb{R}$-linear. We notice that

$$|\lambda_a(x)| \leq \sum_{n=0}^{+\infty} |a_n||x_n| \leq \|a\|_{\ell^\infty} \|x\|_{\ell^1} < +\infty, \qquad \forall\, x = (x_n)_{n\in\mathbb{N}} \in \ell^1.$$

This immediately implies that $\lambda_a \in (\ell^1)^*$ is well-defined and $\|\lambda_a\|_{(\ell^1)^*} \leq \|a\|_{\ell^\infty}$. Consider the sequence $(e^{(k)})_{k\in\mathbb{N}} \subset \ell^1$ given as in point (a). Notice that $\|e^{(k)}\|_{\ell^1} = 1$ for every $k \in \mathbb{N}$ and

$$\|\lambda_a\|_{(\ell^1)^*} \geq \sup_{k\in\mathbb{N}} \lambda_a(x^{(k)}) = \sup_{k\in\mathbb{N}}|a_k| = \|a\|_{\ell^\infty}.$$

Hence, we conclude $\|\lambda_a\|_{(\ell^1)^*} = \|a\|_{\ell^\infty}$.

We define the linear map $\tilde{I} : \ell^\infty \to (\ell^1)^*$ given by $\tilde{I}(a) := \lambda_a$, for every $a \in \ell^\infty$. From what we have proved so far, it follows that $\tilde{I}$ is a linear isometry. In order to conclude, we just need to show that $\tilde{I}$ is surjective. Indeed, pick any $\lambda \in (\ell^1)^*$ and define $a = (a_n)_{n\in\mathbb{N}}$ by $a_n := \lambda(e^{(n)})$ with $(e^{(n)})_{n\in\mathbb{N}} \subset \ell^1$ given as in point (a). Notice that, for every given $n \in \mathbb{N}$, we have

$$|a_n| = |\lambda(e^{(n)})| \leq \|\lambda\|_{(\ell^1)^*},$$

By taking the supremum over $n \in \mathbb{N}$ in the previous inequality we get

$$\|a\|_{\ell^\infty} \leq \|\lambda\|_{(\ell^1)^*} < +\infty,$$

i.e. $a \in \ell^\infty$. In order to conclude, we claim that $I(a) = \lambda_a = \lambda$. Indeed, first notice that for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$ we have

$$\left\| \sum_{n=0}^{k} x_n e^{(n)} - x \right\|_{\ell^1} = \sum_{n=k+1}^{+\infty} |x_n| \to 0 \qquad (k \to +\infty).$$

Hence, by continuity and linearity of $\lambda$ we get

$$\begin{aligned}\lambda(x) &= \lambda\left( \lim_{k\to+\infty} \sum_{n=0}^{k} x_n e^{(n)} \right) = \lim_{k\to+\infty} \lambda\left( \sum_{n=0}^{k} x_n e^{(n)} \right) \\ &= \lim_{k\to+\infty} \sum_{n=0}^{k} x_n \lambda(e^{(n)}) = \lim_{k\to+\infty} \sum_{n=0}^{k} a_n x_n = \lambda_a(x), \qquad \forall\, x = (x_n)_{n\in\mathbb{N}} \in \ell^1.\end{aligned}$$

The statement follows.

(c) We define the linear functional $\lim : c \subset \ell^\infty \to \mathbb{R}$ given by

$$\lim((a_n)_{n\in\mathbb{N}}) = \lim_{n\to+\infty} a_n \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in c.$$

Moreover, we define sublinear functional $\limsup : \ell^\infty \to \mathbb{R}$ given by

$$\limsup((a_n)_{n\in\mathbb{N}}) = \limsup_{n\to+\infty} a_n \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Since it is straightforward that

$$\lim((a_n)_{n\in\mathbb{N}}) \le \limsup((a_n)_{n\in\mathbb{N}}) \qquad \forall\,(a_n)_{n\in\mathbb{N}} \in c,$$

by Hahn-Banach theorem there exists a linear functional $\lambda : \ell^\infty \to \mathbb{R}$ such that $\lambda|_c = \lim$ and

$$\lambda((a_n)_{n\in\mathbb{N}}) \le \limsup((a_n)_{n\in\mathbb{N}}) \qquad \forall\,(a_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

First, we notice that $\lambda$ is continuous, because

$$\lambda((a_n)_{n\in\mathbb{N}}) \le \limsup((a_n)_{n\in\mathbb{N}}) \le \|(a_n)_{n\in\mathbb{N}}\|_{\ell^\infty} \qquad \forall\,(a_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Moreover, by linearity, we have

$$\lambda((a_n)_{n\in\mathbb{N}}) = -\lambda(-(a_n)_{n\in\mathbb{N}}) \ge -\limsup_{n\to+\infty}(-a_n) = \liminf_{n\to+\infty} a_n \qquad \forall\,(a_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

We are just left to show that $\lambda$ cannot be represented as

$$\lambda((a_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{+\infty} x_n a_n, \qquad \forall\,(a_n)_{n\in\mathbb{N}} \in \ell^\infty$$

for some sequence $(x_n)_{n\in\mathbb{N}} \in \ell^1$. By contradiction, assume that such a sequence $(x_n)_{n\in\mathbb{N}}$ exists. Since $(x_n)_{n\in\mathbb{N}} \in \ell^1$, we have $(x_n)_{n\in\mathbb{N}} \in c_0$ and, since $\lambda$ extends lim, we have

$$\sum_{n=0}^{+\infty} |x_n|^2 = \lambda((x_n)_{n\in\mathbb{N}}) = \lim((x_n)_{n\in\mathbb{N}}) = 0.$$

But this implies that $(x_n)_{n\in\mathbb{N}}$ is the zero sequence and this would mean $\lambda = 0$. This is a contradiction and the statement follows.

$\square$

**Exercise 2.2** Recall that a topological space $X$ is called *separable* if it admits a countable and dense subset $S \subset X$.

(a) Let $X$ be a Banach space. Show that if $X^*$ is separable then $X$ is separable.

(b) Prove that $\ell^\infty$ is not a separable Banach space.

(c) Prove that $\ell^1$ is a separable Banach space.

(d) Prove that $(\ell^\infty)^* \not\cong \ell^1$, i.e. show that there is no surjective isometry $I : \ell^1 \to (\ell^\infty)^*$.

**Solution.**

(a) Let $(\lambda_n)_{n\in\mathbb{N}} \subset X^*$ be dense and countable in the unit sphere of $X^*$. Then, we let $S := (x_n)_{n\in\mathbb{N}} \subset X$ be points such that $\lambda_n(x_n) > \frac{1}{2}$ and $\|x_n\| \leq 1$ for every $n \in \mathbb{N}$. We claim that $X = \overline{\operatorname{span}(S)}$. By contradiction, assume that $X \neq \overline{\operatorname{span}(S)}$ and pick any $x \in X \setminus \overline{\operatorname{span}(S)}$ such that $\|x\| = 1$. Define $\lambda : Y := \operatorname{span}(S \cup \{x\}) = \operatorname{span}(S) \oplus \operatorname{span}(x) \to \mathbb{R}$ by

$$\lambda(v + tx) = t \operatorname{dist}(x, \operatorname{span}(S)), \qquad \forall\, v \in \operatorname{span}(S),\ \forall\, t \in \mathbb{R}.$$

Notice that $\|\lambda\|_{Y^*} \leq 1$. By Hahn-Banach theorem, we can extend $\lambda$ to $\tilde{\lambda} \in X^*$ such that $\|\tilde{\lambda}\|_{X^*} = \|\lambda\|_{Y^*} \leq 1$ and $\tilde{\lambda}|_Y = \lambda$. Since $\tilde{\lambda} \equiv 0$ on $\operatorname{span}(S)$, by continuity we have $\tilde{\lambda} \equiv 0$ on $\overline{\operatorname{span}(S)}$. By density, pick $\lambda_{\tilde{n}}$ such that $\|\lambda_{\tilde{n}} - \tilde{\lambda}\|_{X^*} < \frac{1}{4}$. We have

$$\tilde{\lambda}(x_{\tilde{n}}) \geq |\lambda_{\tilde{n}}(x_{\tilde{n}})| - |\lambda_{\tilde{n}}(x_{\tilde{n}}) - \tilde{\lambda}(x_{\tilde{n}})| \geq \frac{1}{2} - \|\lambda_{\tilde{n}} - \tilde{\lambda}\|_{X^*}\|x_{\tilde{n}}\| > \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0$$

and this contradicts $\tilde{\lambda}(x_{\tilde{n}}) = 0$. The statement follows.

(b) Let $\mathscr{P}(\mathbb{N})$ denote the power set of $\mathbb{N}$. Recall that $\mathscr{P}(\mathbb{N})$ is uncountable. We define the family $\{e_I\}_{I\in\mathscr{P}(\mathbb{N})} \subset \ell^\infty$ by

$$(e_I)_n := \begin{cases} 1 & \text{if } n \in I, \\ 0 & \text{if } n \notin I. \end{cases}$$

Notice that $\|e_I - e_J\|_{\ell^\infty} = 1$, for every $I, J \in \mathscr{P}(\mathbb{N})$ such that $I \neq J$. Hence,

$$\mathscr{B} := \{B(e_I, 1/2)\}_{I\in\mathscr{P}(\mathbb{N})}$$

is an uncountably infinite collection of disjoint open balls in $\ell^\infty$. Now let $S$ be any dense subset of $\ell^\infty$. By definition of dense subset, any ball in $\mathscr{B}$ must contain at least one element of $S$. Thus, $S$ must be uncountable. The statement follows.

(c) Let $c_c := \{(a_n)_{n\in\mathbb{N}} \subset \mathbb{R} \text{ s.t. } \exists\, N \in \mathbb{N} : a_n = 0, \forall\, n \in \mathbb{N} \text{ with } n \geq N\} \subset \ell^1$. We claim that $\overline{c_c}^{\|\cdot\|_{\ell^1}} = \ell^1$. Indeed, let $a \in \ell^1$ and consider the sequence $(a^{(k)})_{k\in\mathbb{N}} \subset c_c$ given by

$$a_n^{(k)} := \begin{cases} a_n & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases}$$

for every $k, n \in \mathbb{N}$. We have

$$\|a^{(k)} - a\|_{\ell^1} = \sum_{n=k+1}^{+\infty} |a_n| \to 0 \qquad (k \to +\infty).$$

Our claim follows.

It's not hard to see that

$$c_c^{\mathbb{Q}} := \{(a_n)_{n\in\mathbb{N}} \subset \mathbb{Q} \text{ s.t. } \exists\, N \in \mathbb{N} : a_n = 0, \forall\, n \in \mathbb{N} \text{ with } n \geq N\} \subset c_c \subset \ell^1$$

is dense in $c_c$. We conclude that $c_c^{\mathbb{Q}}$ is dense in $\ell^1$. Since $c_c^{\mathbb{Q}}$ is clearly countable, our statement follows.

(d) It's easy to see that if $X, Y$ are isometrically isomorphic vector spaces, then $X$ is separable if and only if $Y$ is separable. By contradiction, assume that $(\ell^\infty)^* \cong \ell^1$. Since $\ell^1$ is separable (by point (c)), we conclude that $(\ell^\infty)^*$ is separable. Then by point (a) we get that $\ell^\infty$ is separable. But this contradicts point (b) and our statement follows.

□

**Exercise 2.3** Show that the subspaces

$$U := \{(a_n)_{n\in\mathbb{N}} \in \ell^1 \text{ s.t. } a_{2n} = 0, \forall\, n \in \mathbb{N}\}$$
$$V := \{(a_n)_{n\in\mathbb{N}} \in \ell^1 \text{ s.t. } a_{2n-1} = na_{2n}, \forall\, n \in \mathbb{N} \setminus \{0\}\}$$

are both closed in $\ell^1$ but $U \oplus V$ is **not** closed in $\ell^1$.

**Solution.** First we claim that $U$ is closed. Let $(a^{(k)})_{k\in\mathbb{N}} \subset U$ be a sequence such that $a^{(k)} \to a$ as $k \to +\infty$ w.r.t $\|\cdot\|_{\ell^1}$ for some $a \in \ell^1$. Given any $n \in \mathbb{N}$, we have

$$|a_{2n}| = |a_{2n}^{(k)} - a_{2n}| \leq \|a^{(k)} - a\|_{\ell^1}, \qquad \forall\, k \in \mathbb{N}.$$

By letting $k \to +\infty$ in the previous inequality we get $a_{2n} = 0$. By arbitrariness of $n \in \mathbb{N}$, we have $a \in U$ and we conclude that $U$ is closed.

Analogously, we claim that $V$ is closed. Indeed, let $(a^{(k)})_{k\in\mathbb{N}} \subset V$ be a sequence such that $a^{(k)} \to a$ as $k \to +\infty$ w.r.t $\|\cdot\|_{\ell^1}$ for some $a \in \ell^1$. Given any $n \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned}|a_{2n-1} - na_{2n}| &= |(a_{2n-1}^{(k)} - na_{2n}^{(k)}) - (a_{2n-1} - na_{2n})| \\ &\leq |a_{2n-1}^{(k)} - a_{2n-1}| + n|a_{2n}^{(k)} - a_{2n}| \leq (1+n)\|a^{(k)} - a\|_{\ell^1}, \qquad \forall\, k \in \mathbb{N}.\end{aligned}$$

By letting $k \to +\infty$ in the previous inequality we get $a_{2n-1} = na_{2n}$. By arbitrariness of $n \in \mathbb{N} \setminus \{0\}$, we have $a \in V$ and we conclude that $V$ is closed.

We claim that $c_c \subset U \oplus V$. Indeed, let $a \in c_c$ and let $u^a \in U$ be given by

$$u^a_m := \begin{cases} a_m - na_{m+1} & \text{if } m = 2n-1 \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

We have that $v^a := a - u^a$ belongs to $V$, because

$$v^a_{2n-1} - nv^a_{2n} = (a_{2n-1} - u^a_{2n-1}) - n(a_{2n} - u^a_{2n}) = (a_{2n-1} - a_{2n-1} + na_{2n}) - na_{2n} = 0$$

for every $n \in \mathbb{N} \setminus \{0\}$. Our claim follows.

We are ready to get our statement. By contradiction, assume that $U \oplus V$ is closed. By what we have proved so far (see also the solution of Exercise 2.2(c) for the proof of the first equality), we have

$$\ell^1 = \overline{c_c}^{\|\cdot\|_{\ell^1}} \subset \overline{U \oplus V} = U \oplus V \subset \ell^1,$$

which implies $U \oplus V = \ell^1$. But this is false, because the sequence $x = (x_m)_{m\in\mathbb{N}}$ given by

$$x_m := \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \dfrac{1}{n^2} & \text{if } m = 2n \end{cases}$$

belongs to $\ell^1$ and does not belong to $U \oplus V$. Indeed, by contradiction, assume that $x = u + v$ with $u \in U, v \in V$. Then we have

$$v_{2n} = x_{2n} - u_{2n} = \frac{1}{n^2}, \qquad \forall\, n \in \mathbb{N}.$$

But since $v \in V$ it holds that

$$v_{2n-1} = nv_{2n} = \frac{1}{n}, \qquad \forall\, n \in \mathbb{N} \setminus \{0\}.$$

This implies $v \notin \ell^1$ and produces a contradiction. The statement follows. □

**Exercise 2.4** Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Prove that a linear functional $\lambda : X \to \mathbb{K}$ is continuous if and only if $\ker(\lambda)$ is a closed vector subspace of $X$.

(b) Prove that if $V \subset X$ is a closed vector subspace of $X$ and $W \supset V$ is a vector subspace of $X$ such that $W/V$ is finite dimensional, then $W$ is closed.

**Solution.**

(a) Assume that $\lambda$ is continuous. Then, $\ker(\lambda) := \lambda^{-1}(0)$ is closed because it is the preimage of a closed set under a continuous map.

Conversely, assume that $\ker(\lambda)$ is closed. Then, $X/\ker(\lambda)$ is a finite dimensional (because it is isomorphic to $\mathrm{Im}(\lambda) \subset \mathbb{R}$) normed vector space and the quotient map $\pi : X \to X/\ker(\lambda)$ is continuous. Moreover, the map $\tilde{\lambda} : X/\ker(\lambda) \to \mathbb{K}$ given by $\tilde{\lambda}([x]) = \lambda(x)$, for every $[x] = x + \ker(\lambda) \in X/\ker(\lambda)$ is well-defined and linear. Hence, $\tilde{\lambda}$ is continuous because it is a linear map between finite-dimensional vector spaces. Since $\lambda = \tilde{\lambda} \circ \pi$, we conclude that $\lambda$ is continuous as composition of continuous maps.

(b) Since $V$ is closed, then $X/V$ is a normed vector space and the quotient map $\pi : X \to X/V$ is continuous. Notice that $W = \pi^{-1}(W/V)$. Since $W/V \subset X/V$ is a finite-dimensional vector subspace, we have that $W/V$ is a closed subset of $X/V$. Hence, $W$ is closed because it is the preimage of a closed set under a continuous map.

□