Exercise 3.1 Let $(X, \|\cdot\|_X)$ be a Banach space and assume that $U \subset X$ is a closed vector subspace of X. We say that U is *topologically complemented in* X if there exist a vector subspace $V \subset X$ such that the linear isomorphism $I : U \times V \to X$ given by $I(u, v) := u + v$ for very $(u, v) \in U \times V$ is a continuous isomorphism of normed vector spaces with continuous inverse. Recall that the natural norm on $U \times V$ is given by $\|(u, v)\|_{U\times V} := \|u\|_X + \|v\|_X$.

(a) Prove that if $\dim(U) < +\infty$, then U is topologically complemented.

(b) Prove that if $\dim(X/U) < +\infty$, then U is topologically complemented.

Exercise 3.2 By using the Baire category theorem, prove the following statements.

(a) Let $f \in C^0([0, +\infty))$ be such that

$$\lim_{n\to+\infty} f(nt) = 0, \qquad \forall\, t \in [0, +\infty).$$

Prove that

$$\lim_{t\to+\infty} f(t) = 0.$$

(b) Let X be a Banach space. Show that any algebraic basis of X is either finite or uncountable.

(c) Let (X, d) be a complete metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous and real-valued functions on X. Assume that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some function f, i.e.

$$\lim_{n\to+\infty} f_n(x) = f(x), \qquad \forall\, x \in X.$$

Prove that the set of the continuity points of f, given by

$$C = \{x \in X \text{ s.t. } f \text{ is continuous at } x\},$$

is residual and dense in X.

Exercise 3.3 Let $\Omega \subset \mathbb{R}^n$ be a bounded and open subset of $\mathbb{R}^n$ and let $T \in L(L^2(\Omega))$. Assume that $T(u) \in C^0(\overline{\Omega})$ whenever $u \in C^0(\overline{\Omega})$. Prove that T restricts to a bounded linear operator from $C^0(\overline{\Omega})$ onto itself.

Last modified: 7 October 2022

Exercise 3.4 Let

$$X := \{f \in C^0([0, 2\pi]) \text{ s.t. } f(0) = f(2\pi)\} \subset C^0([0, 2\pi])$$

equipped with the usual sup norm.

(a) Show that X is Banach space.

(b) For every $k \in \mathbb{N}$, let $s_k : X \to \mathbb{R}$ be given by

$$s_k(f) := \frac{1}{2\pi} \sum_{n=-k}^{n=k} \int_0^{2\pi} f(t)e^{-int}\,dt, \qquad \forall\, f \in X.$$

Prove that s_k is a bounded linear functional on X for every $k \in \mathbb{N}$.

(c) Show that

$$\sup_{k\in\mathbb{N}} \|s_k\|_{L(X,\mathbb{R})} = +\infty.$$

Hint. Notice that

$$\sum_{n=-k}^{n=k} e^{int} = \frac{e^{i(k+1)t} - e^{ikt}}{e^{it} - 1} = \frac{\sin((k + \frac{1}{2})t)}{\sin(\frac{t}{2})}, \qquad \forall\, t \in (0, 2\pi),\ \forall\, k \in \mathbb{N}.$$

(d) Prove that for every $t \in [0, 2\pi]$ there exists a continuous 2π-periodic function whose Fourier series does not converge at t.

Hint. Use the following equivalent formulation of the Banach-Steinhaus theorem, called *condensation of singularities*: if $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ are Banach spaces and the family $(A_k)_{k\in\mathbb{N}} \subset L(X, Y)$ is such that

$$\sup_{k\in\mathbb{N}} \|A_k\|_{L(X,Y)} = +\infty,$$

then there exists $x \in X$ such that

$$\sup_{k\in\mathbb{N}} \|A_k(x)\|_Y = +\infty.$$