Exercise 3.1 Let $(X, \|\cdot\|_X)$ be a Banach space and assume that $U \subset X$ is a closed vector subspace of X. We say that U is *topologically complemented in* X if there exist a vector subspace $V \subset X$ such that the linear isomorphism $I : U \times V \to X$ given by $I(u, v) := u + v$ for very $(u, v) \in U \times V$ is a continuous isomorphism of normed vector spaces with continuous inverse. Recall that the natural norm on $U \times V$ is given by $\|(u, v)\|_{U\times V} := \|u\|_X + \|v\|_X$.

(a) Prove that if $\dim(U) < +\infty$, then U is topologically complemented.

(b) Prove that if $\dim(X/U) < +\infty$, then U is topologically complemented.

Solution. First, we claim that U is topologically complemented if and only if there exists $P \in L(X)$ such that $P^2 = P$ and $P(X) = U$.

Suppose $U \subset X$ is topologically complemented by $V \subset X$. Then, $I : U \times V \to X$ with $(u, v) \mapsto u + v$ is an continuous isomorphism with continuous inverse. We define

$$\begin{aligned} P_1 \colon U \times V &\to U \times V, \\ (u, v) &\mapsto (u, 0) \end{aligned} \qquad\qquad P := I \circ P_1 \circ I^{-1} \colon X \to X.$$

P_1 is linear, bounded since $\|P_1(u, v)\|_{U\times V} = \|u\|_U \leq \|(u, v)\|_{U\times V}$ and hence continuous. As composition of linear continuous maps, P is linear and continuous. Moreover,

$$\begin{aligned} &P \circ P = (I \circ P_1 \circ I^{-1}) \circ (I \circ P_1 \circ I^{-1}) = I \circ P_1 \circ P_1 \circ I^{-1} = I \circ P_1 \circ I^{-1} = P, \\ &P(X) = I(U \times \{0\}) = U. \end{aligned}$$

Conversely, suppose $U \subset X$ allows a continuous linear map $P \colon X \to X$ with $P \circ P = P$ and $P(X) = U$. Let $V := \ker(P)$. Then

$$P \circ (1 - P) = P - P = 0 \qquad\qquad \Rightarrow (1 - P)(X) \subseteq \ker(P) = V. \tag{1}$$

In fact, $(1 - P)(X) = V$ since given $v \in V$ we have $v = (1 - P)v$. Analogously,

$$(1 - P) \circ P = P - P = 0 \qquad\qquad \Rightarrow\ U = P(X) \subseteq \ker(1 - P). \tag{2}$$

In fact, $U = \ker(1 - P)$ since $x - Px = 0$ implies $x = Px \in U$. We now claim that the map

$$I \colon U \times V \to X, \quad I(u, v) = u + v$$

is continuous and has a continuous inverse. Continuity of I follows directly from

$$\|I(u, v)\|_X = \|u + v\|_X \leq \|u\|_X + \|v\|_X = \|(u, v)\|_{U\times V}.$$

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By the assumptions on P, especially (1), the map

$$\Phi\colon X \to U \times V, \quad \Phi(x) = \big(Px, (1-P)x\big)$$

is well-defined and continuous. Since $Pu = u$ for all $u \in U$ by (2) we have

$$\begin{aligned}(\Phi \circ I)(u,v) &= \Phi(u+v) = \Big(Pu + Pv, u - Pu + v - Pv\Big) = (u,v),\\ (I \circ \Phi)(x) &= I(Px, (1-P)x) = Px + (1-P)x = x,\end{aligned}$$

so Φ is inverse to I. Consequently, U is topologically complemented.

(a) It is sufficient to construct a projection map P as above. Let $e_1, \ldots, e_n$ be a basis of the given finite-dimensional subspace $U \subset X$ let $f_1, \ldots, f_n \in L(U, \mathbb{R})$ be the associated dual basis, uniquely defined by the conditions

$$f_i(e_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

From the Hahn-Banach Theorem it follows that there exist extensions $F_i \in L(X; \mathbb{R})$ with $\|F_i\| = \|f_i\|$. We define

$$P\colon X \to X, \quad P(x) = \sum_{i=1}^n F_i(x)\, e_i.$$

Then P is linear and continuous, since

$$\|Px\|_X \leq \left(\sum_{i=1}^n \|F_i\| \|e_i\|_X\right) \|x\|_X.$$

By construction, $P(X) \subset \operatorname{span}\{e_1, \ldots, e_n\} = U$. By definition of f_i and F_i we have $P(e_i) = e_i$ for every $i \in \{1, \ldots, n\}$. Therefore, $P(X) = U$. Finally, for every $x \in X$,

$$(P \circ P)(x) = P\Big(\sum_{i=1}^n F_i(x)\, e_i\Big) = \sum_{i=1}^n F_i(x)\, P(e_i) = \sum_{i=1}^n F_i(x)\, e_i = P(x).$$

It follows from Exercise 4.3 that U is topologically complemented.

(b) Denote by $\pi\colon X \to X/U$, $\pi(x) = [x]$ the canonical quotient map. Since $\dim(X/U) = m < \infty$ we can choose a basis $[e_1], \ldots, [e_m]$ for X/U and let as above $f_1, \ldots f_m \in L(X/U, \mathbb{R})$ be the associated dual basis. Set $F_i := f_i \circ \pi\colon X \to \mathbb{R}$ and define

$$P\colon X \to X, \quad P(x) = \sum_{i=1}^n F_i(x)\, e_i.$$

Since $F_i(e_j) = f_i(\pi(e_j)) = f_i([e_j]) = \delta_{ij}$ we have $P \circ P = P$ as above. Since $[e_1], \dots, [e_m]$ is a basis for X/U, the representatives $e_1, \dots, e_m$ must be linearly independent in X. Therefore, $P(x) = 0$ implies $F_i(x) = f_i([x]) = 0$ for every $i \in \{1, \dots, n\}$ which in turn implies $[x] = [0]$ or $x \in U$. Conversely, $x \in U$ implies $\pi(x) = [0]$ and $P(x) = 0$. Thus we have shown $\ker(P) = U$. As in Exercise 4.3, we conclude that $(1 - P)$ is a continuous projection onto U which implies that U is topologically complemented. $\square$

Exercise 3.2 By using the Baire category theorem, prove the following statements.

(a) Let $f \in C^0([0, +\infty))$ be such that

$$\lim_{n\to+\infty} f(nt) = 0, \qquad \forall\, t \in [0, +\infty).$$

Prove that

$$\lim_{t\to+\infty} f(t) = 0.$$

(b) Let X be a Banach space. Show that any algebraic basis of X is either finite or uncountable.

(c) Let (X, d) be a complete metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous and real-valued functions on X. Assume that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some function f, i.e.

$$\lim_{n\to+\infty} f_n(x) = f(x), \qquad \forall\, x \in X.$$

Prove that the set of the continuity points of f, given by

$$C = \{x \in X \text{ s.t. } f \text{ is continuous at } x\},$$

is residual and dense in X.

Solution.

(a) Define $f_n(t) = |f(nt)|$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and let

$$A_N := \bigcap_{n=N}^{\infty} \{t \in [0, \infty) \mid f_n(t) \le \varepsilon\}.$$

Since f_n is continuous, the preimage $f_n^{-1}([0, \varepsilon]) = \{t \in [0, \infty) \mid f_n(t) \le \varepsilon\}$ is closed for all $n \in \mathbb{N}$. Thus, the set A_N is closed as intersection of closed sets. By assumption,

$$\forall t \in [0, \infty) \quad \exists N_t \in \mathbb{N} \quad \forall n \ge N_t : \quad f_n(t) \le \varepsilon$$

which implies

$$[0,\infty) = \bigcup_{N=1}^{\infty} A_N.$$

Baire's Lemma applied to the complete metric space $([0,\infty), |\cdot|)$ implies that there exists $N_0 \in \mathbb{N}$ such that A_{N_0} has non-empty interior, i. e. there exist $0 \leq a < b$ such that $(a,b) \subset A_{N_0}$. This implies

$$\begin{aligned} & \forall n \geq N_0 \quad \forall t \in (a,b): \qquad f_n(t) \leq \varepsilon \\ \Leftrightarrow \quad & \forall n \geq N_0 \quad \forall t \in (na,nb): \ |f(t)| \leq \varepsilon. \end{aligned}$$

If $n > \frac{a}{b-a}$, then $(n+1)a < nb$. For the intervals $J_{a,b}(n) := (na,nb)$ this means that $J_{a,b}(n) \cap J_{a,b}(n+1) \neq \emptyset$. Let $N_1 > \max\{N_0, \frac{a}{b-a}\}$. Then, in particular,

$$\forall t > N_1 a: \qquad |f(t)| \leq \varepsilon.$$

This proves $\lim\limits_{t\to\infty} f(t) = 0$ since $\varepsilon > 0$ was arbitrary.

(b) Assume by contradiction that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$. For $n \in \mathbb{N}$ we define the linear subspaces $A_n = \operatorname{span}\{e_1, \ldots, e_n\} \subset X$.

As finite dimensional subspace, A_n is closed. Suppose that A_n has non-empty interior. Then there exist $x \in A_n$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_n$. Since A_n is a linear subspace, we may subtract $x \in A_n$ from the elements in $B_\varepsilon(x)$ to obtain $B_\varepsilon(0) \subset A_n$. For the same reason,

$$A_n \supset \{\lambda y \mid \lambda > 0, \ y \in B_\varepsilon(x)\} = X.$$

This implies $\dim X \leq n$ which contradicts our assumption that the algebraic basis of X is infinite. Thus A_n must have empty interior and thus, being also closed, is nowhere dense. By assumption,

$$X = \bigcup_{n\in\mathbb{N}} A_n,$$

which implies that X is meager. Since X is complete, this contradicts Baire's Lemma.

(c) First we notice that

$$C = \Big\{ x \in X \ \text{ s.t. } \operatorname{osc}_x(f) := \lim_{r\to 0^+} \Big(\sup_{y\in B_r(x)} f(y) - \inf_{y \in B_r(x)} f(y) \Big) = 0 \Big\}.$$

By letting

$$D_\varepsilon := \{x \in X \ \text{ s.t. } \operatorname{osc}_x(f) \geq \varepsilon\} \quad \forall\, \varepsilon > 0,$$

we can write

$$C = \bigcap_{\substack{j\in\mathbb{N} \\ j\geq 1}} D_{1/j}^c.$$

By the Baire category theorem, in order to conclude it is sufficient to show that D_ε^c is open and dense for every $\varepsilon > 0$.

Openness. We want to show that D_ε is closed for every $\varepsilon > 0$. Fix any $\varepsilon > 0$. We notice that D_ε is a super lever set of the function $X \ni x \mapsto \operatorname{osc}_x(f)$. Hence, we just need to show that $X \ni x \mapsto \operatorname{osc}_x(f)$ is upper semicontinuous. First, we show that the function $g : X \to \mathbb{R}$ given by

$$g(x) := \lim_{r\to 0^+} \sup_{y\in B_r(x)} f(y) \qquad \forall\, x \in X$$

is upper semicontinuous. Fix any $x \in X$. By definition of g, for every $\eta > 0$ there exists $\delta > 0$ such that $f(y) \leq g(x) + \eta$, for every $y \in B_\delta(x)$. Therefore, for every $y \in B_{\delta/2}(x)$ we have

$$g(y) = \lim_{r\to 0^+} \sup_{z\in B_r(y)} f(z) \leq \sup_{z\in B_{\delta/2}(y)} f(z) \leq g(x) + \eta$$

since $B_{\delta/2}(y) \subset B_\delta(x)$. By taking the limit superior as $y \to x$ in the previous inequality we get

$$\limsup_{y\to x} g(y) \leq g(x) + \eta$$

and by arbitrariness of $\eta >$) we get that g is upper semicontinuous. Now we consider the function $h : X \to \mathbb{R}$ given by

$$h(x) := -\lim_{r\to 0^+} \inf_{y\in B_r(x)} f(y) = \lim_{r\to 0^+} \sup_{y\in B_r(x)} \{-f(y)\} \qquad \forall\, x \in X$$

and we notice that applying exactly the same argument with $-f$ instead of f we get that h is upper semicontinuous as well. Since $x \mapsto \operatorname{osc}_x(f) = g + h$, we get that $x \mapsto \operatorname{osc}_x(f)$ is upper semicontinuous and we are done.

Density. Fix any $\varepsilon > 0$. For every $k \in \mathbb{N}$ we define

$$E_k := \bigcap_{i,j\geq k} \{x \in X \text{ s.t. } |f_i(x) - f_j(x)| \leq \varepsilon/4\}.$$

Notice that E_k is closed as intersection of closed sets (recall that each f_n is continuous) for every $k \in \mathbb{N}$. Moreover,

$$X = \bigcup_{k\in\mathbb{N}} E_k$$

because the functions $(f_n)_{n\in\mathbb{N}}$ pointwise converges to f. As a result, by the Baire category theorem, for every open set $U \subset X$ there exists $k \in \mathbb{N}$ such that $E_k^\circ \cap U \neq \emptyset$. In particular, there exists an open set $V \subset E_k \cap U$. Hence, by definition of E_k, we have $|f_i(x) - f_j(x)| \leq \varepsilon/4$ for every $x \in V$ and $i, j \geq k$. Taking $i = k$ and letting $j \to +\infty$ we get $|f_k(x) - f(x)| \leq \varepsilon/4$, for every $x \in V$. Since f_k is continuous, by taking V possibily smaller we can assume that $|f_k(x) - f_k(y)| \leq \varepsilon/4$, for every $x, y \in V$. In conclusion, for every $x, y \in V$ we get

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \leq \frac{3\varepsilon}{4},$$

which implies $\operatorname{osc}_x(f) \leq 3\varepsilon/4 < \varepsilon$ for every $x \in V$. Thus, $V \subset D_\varepsilon^c$ and $U \cap D_\varepsilon^x \neq \emptyset$. By arbitrariness of the open set $U \subset X$, our claim follows.

□

Exercise 3.3 Let $\Omega \subset \mathbb{R}^n$ be a bounded and open subset of $\mathbb{R}^n$ and let $T \in L(L^2(\Omega))$. Assume that $T(u) \in C^0(\overline{\Omega})$ whenever $u \in C^0(\overline{\Omega})$. Prove that T restricts to a bounded linear operator from $C^0(\overline{\Omega})$ onto itself.

Solution. Denote by $S : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ the restriction of T to $C^0(\overline{\Omega})$. Such restriction is a well-defined linear map by hypothesis. By the closed graph theorem, to prove that S is bounded if is enough to show that its graph is closed. Let $(f_n, S(f_n))_{n\in\mathbb{N}}$ be any sequence in the graph of S such that $(f_n, S(f_n)) \to (f, g) \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$. Since $f_n \to f$ and $S(f_n) = T(f_n) \to g$ as $n \to +\infty$ both w.r.t. the sup norm on $C^0(\overline{\Omega})$, we have

$$\begin{aligned} \|f_n - f\|_{L^2(\Omega)} &\leq |\Omega|^{1/2} \|f_n - f\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } n \to +\infty, \\ \|T(f_n) - g\|_{L^2(\Omega)} &\leq |\Omega|^{1/2} \|S(f_n) - g\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } n \to +\infty. \end{aligned}$$

Hence, $(f_n, T(f_n)) \to (f, g)$ w.r.t to the norm on $L^2(\Omega) \times L^2(\Omega)$. By the closed graph theorem, since $T \in L(L^2(\Omega))$ its graph is closed. Hence, $g = T(f) = S(f)$ in $L^2(\Omega)$, i.e almost everywhere on Ω. But since both g and $S(f)$ are continuous up to the boundary of Ω, we conclude that $g = S(f)$ on all of $\overline{\Omega}$ and the statement follows. □

Exercise 3.4 Let

$$X := \{f \in C^0([0, 2\pi]) \text{ s.t. } f(0) = f(2\pi)\} \subset C^0([0, 2\pi])$$

equipped with the usual sup norm.

(a) Show that X is Banach space.

(b) For every $k \in \mathbb{N}$, let $s_k : X \to \mathbb{R}$ be given by

$$s_k(f) := \frac{1}{2\pi} \sum_{n=-k}^{n=k} \int_0^{2\pi} f(t) e^{-int}\, dt, \qquad \forall\, f \in X.$$

Prove that s_k is a bounded linear functional on X for every $k \in \mathbb{N}$.

(c) Show that

$$\sup_{k \in \mathbb{N}} \|s_k\|_{L(X,\mathbb{R})} = +\infty.$$

Hint. Notice that

$$\sum_{n=-k}^{n=k} e^{int} = \frac{e^{i(k+1)t} - e^{ikt}}{e^{it} - 1} = \frac{\sin((k+\frac{1}{2})t)}{\sin(\frac{t}{2})}, \qquad \forall\, t \in (0, 2\pi),\ \forall\, k \in \mathbb{N}.$$

(d) Prove that for every $t \in [0, 2\pi]$ there exists a continuous 2π-periodic function whose Fourier series does not converge at t.

Hint. Use the following equivalent formulation of the Banach-Steinhaus theorem, called *condensation of singularities*: if $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ are Banach spaces and the family $(A_k)_{k\in\mathbb{N}} \subset L(X,Y)$ is such that

$$\sup_{k \in \mathbb{N}} \|A_k\|_{L(X,Y)} = +\infty,$$

then there exists $x \in X$ such that

$$\sup_{k \in \mathbb{N}} \|A_k(x)\|_Y = +\infty.$$

Solution.

(a) Since X is a linear subspace of the Banach space $C^0([0, 2\pi])$, in order to show that X is Banach it is enough to show that X is a closed subspace. Indeed, let $(f_n)_{n\in\mathbb{N}} \subset X$ be such that $f_n \to f \in C^0([0, 2\pi])$ as $n \to +\infty$ w.r.t. the sup norm. We need to show that $f \in X$. Since $f_n \to f$ pointwise on $[0, 2\pi]$ and $f_n(0) = f_n(2\pi) = 0$ for every $n \in \mathbb{N}$, we conclude that $f(0) = f(2\pi) = 0$ and the statement follows.

(b) Fix any $k \in \mathbb{N}$. Since the linearity of s_k is clear, we are just left to show its continuity. We compute

$$|s_k(f)| \le \frac{1}{2\pi} \sum_{n=-k}^{n=k} \int_0^{2\pi} |f(t)|\, dt \le (2k+1)\|f\|_\infty, \quad \forall\, f \in X.$$

In particular then $\|s_k\|_{L(X,\mathbb{R})} \le 2k+1$ for every $k \in \mathbb{N}$.

(c) By using the hint, we get that for every $k \in \mathbb{N}$ and $f \in X$ we have

$$s_k(f) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin((k+\frac{1}{2})t)}{\sin(\frac{t}{2})} f(t)\, dt.$$

Notice that for every $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin((k+\frac{1}{2})t)}{\sin(\frac{t}{2})} \right| dt &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|\sin((k+\frac{1}{2})t)|}{\pi t}\, dt = \int_0^{(2k+1)\pi} \frac{|\sin(u)|}{\pi u}\, du \\
&\geq \sum_{n=0}^{k} \int_{2n\pi+\frac{\pi}{6}}^{2n\pi+\frac{5\pi}{6}} \frac{|\sin(u)|}{\pi u}\, du \\
&\geq \sum_{n=0}^{2k} \int_{2n\pi+\frac{\pi}{6}}^{2n\pi+\frac{5\pi}{6}} \frac{1}{2\pi u}\, du \geq \sum_{n=0}^{2k} \int_{2n\pi+\frac{\pi}{3}}^{2n\pi+\frac{2\pi}{3}} \frac{1}{2\pi(n+1)}\, du \\
&\geq \sum_{n=0}^{2k} \frac{1}{3(n+1)} = \frac{1}{3} \sum_{n=1}^{2k+1} \frac{1}{n}.
\end{aligned}$$

This implies that

$$\sup_{k\in\mathbb{N}} \|s_k\|_{L(X,\mathbb{R})} \geq \frac{1}{3} \sup_{k\in\mathbb{N}} \sum_{n=1}^{2k+1} \frac{1}{n} = \frac{1}{3} \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty.$$

(d) By using the hint, we get that there exists $f \in X$ such that

$$\sup_{k\in\mathbb{N}} |s_k(f)| = +\infty.$$

Hence, given any $t \in [0, 2\pi]$ the function $[0, 2\pi] \ni s \to f(s-t)$ is a continuous 2π-periodic function whose partial Fourier sums don't converge at t.

□