Exercise 4.1 For $f \in L^1([0,2\pi],\mathbb{C})$ we define the *k-th Fourier coefficient* of f to be

$$\hat{f}(k) := \int_0^{2\pi} f(t)e^{-ikt}\,dt \qquad \forall\, k \in \mathbb{Z}$$

and we let $\mathcal{F}(f) := (\hat{f}(k))_{k\in\mathbb{Z}}$.

(a) Show that $\mathcal{F} : L^1([0,2\pi],\mathbb{C}) \to \ell^\infty(\mathbb{Z})$ is a bounded linear operator.

(b) Prove the Riemann-Lebesgue lemma, i.e. show $\mathcal{F}(f) \in c_0(\mathbb{Z})$ for every $f \in L^1(0,2\pi)$.

(c) Prove that $\mathcal{F} : L^1([0,2\pi],\mathbb{C}) \to c_0(\mathbb{Z})$ has dense range but it is not surjective.

Hint. Use the open mapping theorem to solve part (c).

Exercise 4.2 Let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y_1, \|\cdot\|_{Y_1})$, $(Y_2, \|\cdot\|_{Y_2}), \ldots$ be normed spaces. For every $n \in \mathbb{N}$, let $G_n \subset L(X, Y_n)$ be an unbounded set of linear continuous mappings from X to Y_n. Prove that there exists $x \in X$ such that

$$\sup_{T\in G_n} \|Tx\|_{Y_n} = +\infty, \qquad \forall\, n \in \mathbb{N}.$$

Hint. Use the Baire category theorem and the Banach-Steinhaus theorem.

Exercise 4.3 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(a) Show that B is continuous if

$$\exists C > 0 \quad \forall (x,y) \in X \times Y : \quad \|B(x,y)\|_Z \le C\|x\|_X\|y\|_Y. \tag{†}$$

(b) Assume that $(X, \|\cdot\|_X)$ is Banach. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x,y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x',y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then B is continuous.

Hint. Part (b) uses the Banach-Steinhaus theorem.

Exercise 4.4 Let $\mathbb{D} := \{z \in \mathbb{C} \text{ s.t. } |z| < 1\} \subset \mathbb{C}$ and let

$$\mathcal{H}^2(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ s.t. } f \text{ is holomorphic with } \sup_{0\le r<1} \int_0^{2\pi} |f(re^{it})|^2\,dt < +\infty \right\}.$$

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(a) Derive a characterization of all the functions $f \in \mathcal{H}^2(\mathbb{D})$ in terms of the coefficients $(a_k(f))_{k\in\mathbb{N}} \subset \mathbb{C}$ of their power series expansion, i.e. those coefficients for which

$$f(z) = \sum_{k=0}^{+\infty} a_k(f) z^k, \qquad \forall\, z \in \mathbb{D}.$$

(b) Prove that, for every $f, g \in \mathcal{H}^2(\mathbb{D})$, the limit

$$\langle f, g\rangle := \frac{1}{2\pi} \lim_{r\to 1^-} \int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\, dt$$

exists and express it in terms of the coefficients $(a_k(f))_{k\in\mathbb{N}}$, $(a_k(g))_{k\in\mathbb{N}} \subset \mathbb{C}$ of their power series expansions.

(c) Prove that $(\mathcal{H}^2(\mathbb{D}), \langle\cdot,\cdot\rangle)$ is a Hilbert space with $(\mathbb{D} \ni z \mapsto z^n \in \mathbb{C})_{n\in\mathbb{N}}$ being an orthonormal basis.