Exercise 4.1 For $f \in L^1([0,2\pi],\mathbb{C})$ we define the *k-th Fourier coefficient* of f to be

$$\hat{f}(k) := \int_0^{2\pi} f(t)e^{-ikt}\,dt \qquad \forall\, k \in \mathbb{Z}$$

and we let $\mathcal{F}(f) := (\hat{f}(k))_{k\in\mathbb{Z}}$.

(a) Show that $\mathcal{F} : L^1([0,2\pi],\mathbb{C}) \to \ell^\infty(\mathbb{Z})$ is a bounded linear operator.

(b) Prove the Riemann-Lebesgue lemma, i.e. show $\mathcal{F}(f) \in c_0(\mathbb{Z})$ for every $f \in L^1(0,2\pi)$.

(c) Prove that $\mathcal{F} : L^1([0,2\pi],\mathbb{C}) \to c_0(\mathbb{Z})$ has dense range but it is not surjective.

Hint. Use the open mapping theorem to solve part (c).

Solution.

(a) The linearity of $\mathcal{F}$ is clear from the linearity of the integral. Fix any $f \in L^1(0,2\pi)$ and notice that for every $k \in \mathbb{Z}$ we have

$$|\hat{f}(k)| = \left|\int_0^{2\pi} f(t)e^{-ikt}\,dt\right| \le \int_0^{2\pi} |f(t)e^{-ikt}|\,dt = \int_0^{2\pi} |f(t)|\,dt = \|f\|_{L^1([0,2\pi],\mathbb{C})}.$$

Hence, $\|\mathcal{F}(f)\|_{\ell^\infty(\mathbb{Z})} \le \|f\|_{L^1([0,2\pi],\mathbb{C})}$ for every $f \in L^1([0,2\pi],\mathbb{C})$ an we conclude that $\mathcal{F}$ is a bounded linear operator.

(b) First, consider any $f \in C^0([0,2\pi],\mathbb{C})$. Since f is uniformly continuous on $[0,2\pi]$, there exists $\delta : (0,+\infty) \to (0,+\infty)$ such that for every $\varepsilon \in (0,+\infty)$ we have

$$x, y \in [0,2\pi] \text{ and } |x-y| < \delta(\varepsilon) \;\Rightarrow\; |f(x)-f(y)| < \varepsilon.$$

Hence, for every $\varepsilon \in (0,+\infty)$ and for every $k \in \mathbb{N}$ we have

$$\left|\int_0^{2\pi} f(t)e^{-ikt}\,dt - \sum_{m=0}^{\left[\frac{2\pi}{\delta(\varepsilon)}\right]} f(m\delta(\varepsilon)) \int_{m\delta(\varepsilon)}^{\min\{(m+1)\delta(\varepsilon),2\pi\}} e^{-ikt}\,dt\right| \le 2\pi\varepsilon.$$

Moreover, for every $\varepsilon \in (0,+\infty)$ it holds that

$$\limsup_{k\to+\infty}\left|\sum_{m=0}^{\left[\frac{2\pi}{\delta(\varepsilon)}\right]} f(m\delta(\varepsilon)) \int_{m\delta(\varepsilon)}^{\min\{(m+1)\delta(\varepsilon),2\pi\}} e^{-ikt}\,dt\right| \le \limsup_{k\to+\infty} \frac{2}{k}\|f\|_{L^\infty}\left(\left[\frac{2\pi}{\delta(\varepsilon)}\right]+1\right) = 0.$$

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By combining the previous two inequalities, we get

$$\limsup_{k\to+\infty} |\hat{f}(k)| = \limsup_{k\to+\infty} \left| \int_0^{2\pi} f(t)e^{-ikt}\, dt \right| \leq 2\pi\varepsilon, \qquad \forall\, \varepsilon \in (0,+\infty).$$

By letting $\varepsilon \to 0^+$ in the previous inequality we get the statement for $f \in C^0([0,2\pi],\mathbb{C})$.

Next, pick any $f \in L^1([0,2\pi],\mathbb{C})$. Since $C^0([0,2\pi],\mathbb{C})$ is dense in $L^1([0,2\pi],\mathbb{C})$, for every given $\varepsilon > 0$ we can find $g_\varepsilon \in C^0([0,2\pi],\mathbb{C})$ such that $\|f - g_\varepsilon\|_{L^1([0,2\pi],\mathbb{C})} \leq \varepsilon$. Thus, we have that

$$\limsup_{k\to+\infty} |\hat{f}(k)| = \limsup_{k\to+\infty} \left| \int_0^{2\pi} f(t)e^{-ikt}\, dt \right| \leq \limsup_{k\to+\infty} \left| \int_0^{2\pi} g_\varepsilon(t)e^{-ikt}\, dt \right| + \varepsilon \leq \varepsilon,$$

for every $\varepsilon > 0$. By letting $\varepsilon \to 0^+$ in the previous inequality, the statement follows.

(c) Let $(e^{(k)})_{k\in\mathbb{Z}} \subset c_0(\mathbb{Z})$ be given by $e^{(k)}_n := \delta_{nk}$, for every $k,n \in \mathbb{Z}$. Notice that $(e^{(k)})_{k\in\mathbb{Z}}$ is dense in $c_0(\mathbb{Z})$. Moreover, we have $(e^{(k)})_{k\in\mathbb{N}} \subset \mathcal{F}(L^1([0,2\pi],\mathbb{C}))$, because $e^{(k)} = \mathcal{F}([0,2\pi] \ni t \mapsto e^{ikt} \in \mathbb{C})$ for every $k \in \mathbb{Z}$. Hence, we conclude that $\mathcal{F}(L^1([0,2\pi],\mathbb{C}))$ is dense in $c_0(\mathbb{Z})$.

By contradiction, assume that $\mathcal{F}(L^1([0,2\pi],\mathbb{C})) = c_0(\mathbb{Z})$. Notice that $\mathcal{F}$ is injective. Indeed, if $\mathcal{F}(f) = 0$ then we have that

$$\int_0^{2\pi} f(t)p(t)\, dt = 0 \tag{1}$$

for every trigonometric polynomial on $[0,2\pi]$. By the density of the trigonometric polynomials in the 2π-periodic continuous functions on $[0,2\pi]$, we have that (1) holds for every continuous and 2π-periodic function on $[0,2\pi]$. Hence, $f = 0$ by the fundamental principle of the calculus of variations.

Then, by the open mapping theorem we would have that $\mathbb{F}$ is continuously invertible. To see that this is not the case, consider the functions $(f_n)_{n\in\mathbb{N}} \subset L^1([0,2\pi],\mathbb{C})$ given by

$$f_n(t) = \sum_{k=-n}^{k=n} e^{ikt}, \qquad \forall\, t \in [0,2\pi],\, \forall\, n \in \mathbb{N}.$$

Note that for all $n \in \mathbb{N}$ we have

$$f_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})}, \qquad \forall\, t \in [0,2\pi],\, \forall\, n \in \mathbb{N}.$$

Thus, we obtain that

$$\begin{aligned}\|f_n\|_{L^1([0,2\pi],\mathbb{C})} &= \int_0^{2\pi} \frac{|\sin((n+\frac{1}{2})t)|}{|\sin(\frac{t}{2})|}\,dt \geq (2n+1)\int_0^{2\pi} \frac{|\sin((n+\frac{1}{2})t)|}{(n+\frac{1}{2})t}\,dt \\ &= 2\int_0^{(2n+1)\pi} \frac{|\sin(t)|}{t}\,dt.\end{aligned}$$

Since the latter integral diverges to $+\infty$ as $n \to +\infty$, we have $\|f_n\|_{L^1([0,2\pi],\mathbb{C})} \to +\infty$ as $n \to +\infty$. But since $\|\mathcal{F}(f_n)\|_{\ell^\infty(\mathbb{Z})} = 1$, for every $n \in \mathbb{N}$, we conclude that $\mathcal{F}$ cannot be continuously invertible.

□

Exercise 4.2 Let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y_1, \|\cdot\|_{Y_1})$, $(Y_2, \|\cdot\|_{Y_2}), \ldots$ be normed spaces. For every $n \in \mathbb{N}$, let $G_n \subset L(X, Y_n)$ be an unbounded set of linear continuous mappings from X to Y_n. Prove that there exists $x \in X$ such that

$$\sup_{T \in G_n} \|Tx\|_{Y_n} = +\infty, \qquad \forall\, n \in \mathbb{N}.$$

Hint. Use the Baire category theorem and the Banach-Steinhaus theorem.

Solution. Assume by contradiction that for every $x \in X$ there exists $n \in \mathbb{N}$ such that

$$\sup_{T \in G_n} \|Tx\|_{Y_n} < +\infty.$$

Hence, we can write

$$X = \bigcup_{n \in \mathbb{N}} A_n \quad \text{with} \quad A_n := \left\{x \in X \text{ s.t. } \sup_{T \in G_n} \|Tx\|_{Y_n} < +\infty\right\}, \forall\, n \in \mathbb{N}.$$

Notice that X would be meagre if all the A_n, $n \in \mathbb{N}$, were meagre. Since X is not meagre by the Baire category theorem, we have that there exists $N \in \mathbb{N}$ such that A_N is not meagre. Notice that

$$A_N := \bigcup_{k \in \mathbb{N}} C_k \quad \text{with} \quad C_k := \left\{x \in X \text{ s.t. } \sup_{T \in G_N} \|Tx\|_{Y_N} \leq k\right\}, \forall\, k \in \mathbb{N}.$$

Since $G_N \in L(X, Y_N)$, we have that C_k is closed for every $k \in \mathbb{N}$. together with the fact that A_N is not meagre, this implies that there exists $K \in \mathbb{N}$ such that $C_K^\circ \neq \emptyset$. That is, there exist $x \in X$ and $\varepsilon \in (0, +\infty)$ such that $\{y \in X \text{ s.t. } \|y - x\|_X < \varepsilon\} \subset C_K$. This implies that for every $y \in X$ with $\|y\|_X \leq 1$ we have

$$\begin{aligned} \|Ty\|_{Y_N} &= \frac{2}{\varepsilon} \left\| T\left(x + \frac{\varepsilon}{2} y - x\right) \right\|_{Y_N} \\ &\leq \frac{2}{\varepsilon} \left(\left\| T\left(x + \frac{\varepsilon}{2} y\right) \right\|_{Y_N} + \|T(x)\|_{Y_N} \right) \leq \frac{4K}{\varepsilon}, \quad \forall T \in G_N. \end{aligned}$$

This contradict the hypothesis that G_N is unbounded and the statement follows. □

Exercise 4.3 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X \times Y})$, where $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(a) Show that B is continuous if

$$\exists C > 0 \quad \forall (x, y) \in X \times Y : \quad \|B(x, y)\|_Z \leq C\|x\|_X \|y\|_Y. \tag{†}$$

(b) Assume that $(X, \|\cdot\|_X)$ is Banach. Assume further that the maps

$$\begin{aligned} X &\to Z & \qquad Y &\to Z \\ x &\mapsto B(x, y') & \qquad y &\mapsto B(x', y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then B is continuous.

Hint. Part (b) uses the Banach-Steinhaus theorem.

Solution.

(a) Let $((x_k, y_k))_{k \in \mathbb{N}}$ be a sequence in $X \times Y$ converging to (x, y) in $(X \times Y, \|\cdot\|_{X \times Y})$. By definition,

$$\|x_k - x\|_X + \|y_k - y\|_Y = \|(x_k - x, y_k - y)\|_{X \times Y} = \|(x_k, y_k) - (x, y)\|_{X \times Y}$$

which yields convergence $x_k \to x$ in X and $y_k \to y$ in Y. Since $B\colon X \times Y \to Z$ is bilinear, we have

$$\begin{aligned} \|B(x_k, y_k) - B(x, y)\|_Z &= \|B(x_k, y_k) - B(x, y_k) + B(x, y_k) - B(x, y)\|_Z \\ &= \|B(x_k - x, y_k) - B(x, y_k - y)\|_Z \\ &\leq \|B(x_k - x, y_k)\|_Z + \|B(x, y_k - y)\|_Z. \end{aligned}$$

Using the assumption $\|B(x, y)\|_Z \leq C\|x\|_X \|y\|_Y$ and the fact that convergence of $(y_k)_{k \in \mathbb{N}}$ in $(Y, \|\cdot\|_Y)$ implies that $\|y_k\|_Y$ is bounded uniformly for all $k \in \mathbb{N}$, we conclude

$$\|B(x_k, y_k) - B(x, y)\|_Z \leq C\|x - x_k\|_X \|y_k\|_Y + C\|x\|_X \|y - y_k\|_Y \xrightarrow{k \to \infty} 0.$$

(b) Let $B_1^Y \subset Y$ be the unit ball around the origin in $(Y, \|\cdot\|_Y)$. For every $x \in X$ we have by assumption

$$\sup_{y' \in B_1^Y} \|B(x,y')\|_Z \leq \sup_{y' \in B_1^Y} \|y'\|_Y \|B(x,\cdot)\|_{L(Y,Z)} \leq \|B(x,\cdot)\|_{L(Y,Z)} < \infty,$$

which means that the maps $(B(\cdot,y'))_{y' \in B_1^Y} \in L(X,Z)$ are pointwise bounded. Since X is assumed to be complete, the Theorem of Banach-Steinhaus implies that $(B(\cdot,y'))_{y' \in B_1^Y} \in L(X,Z)$ are uniformly bounded, i. e.

$$C := \sup_{y' \in B_1^Y} \|B(\cdot,y')\|_{L(X,Z)} < \infty.$$

From this we conclude

$$\begin{aligned} \|B(x,y)\|_Z &= \|y\|_Y \Big\| B\Big(x, \frac{y}{\|y\|_Y}\Big) \Big\|_Z \\ &\leq \|y\|_Y \|x\|_X \Big\| B\Big(\cdot, \frac{y}{\|y\|_Y}\Big) \Big\|_{L(X,Z)} \leq C \|x\|_X \|y\|_Y, \end{aligned}$$

so B is continuous by (a). □

Exercise 4.4 Let $\mathbb{D} := \{z \in \mathbb{C} \text{ s.t. } |z| < 1\} \subset \mathbb{C}$ and let

$$\mathcal{H}^2(\mathbb{D}) := \Big\{ f : \mathbb{D} \to \mathbb{C} \text{ s.t. } f \text{ is holomorphic with } \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^2 \, dt < +\infty \Big\}.$$

(a) Derive a characterization of all the functions $f \in \mathcal{H}^2(\mathbb{D})$ in terms of the coefficients $(a_k(f))_{k \in \mathbb{N}} \subset \mathbb{C}$ of their power series expansion, i.e. those coefficients for which

$$f(z) = \sum_{k=0}^{+\infty} a_k(f) z^k, \qquad \forall\, z \in \mathbb{D}.$$

(b) Prove that, for every $f, g \in \mathcal{H}^2(\mathbb{D})$, the limit

$$\langle f, g \rangle := \frac{1}{2\pi} \lim_{r \to 1^-} \int_0^{2\pi} f(re^{it}) \overline{g(re^{it})} \, dt$$

exists and express it in terms of the coefficients $(a_k(f))_{k \in \mathbb{N}}$, $(a_k(g))_{k \in \mathbb{N}} \subset \mathbb{C}$ of their power series expansions.

(c) Prove that $(\mathcal{H}^2(\mathbb{D}), \langle \cdot, \cdot \rangle)$ is a Hilbert space with $(\mathbb{D} \ni z \mapsto z^n \in \mathbb{C})_{n \in \mathbb{N}}$ being an orthonormal basis.

Solution.

(a) From complex analysis, we know that for every $f \in \mathcal{H}^2(\mathbb{D})$

$$f(z) = \sum_{k=0}^{+\infty} a_k(f) z^k, \qquad \forall\, z \in \mathbb{D},$$

the convergence radius is at least 1 and that the power series convergences locally uniformly to f. Thus, we obtain that for all $r \in (0,1)$ that

$$\begin{aligned}\int_0^{2\pi} |f(re^{it})|^2 \, dt &= \int_0^{2\pi} \sum_{k,l=0}^{+\infty} a_k(f)\overline{a_l(f)} r^{k+l} e^{i(k-l)t} \, dt \\ &= 2\pi \sum_{k,l=0}^{+\infty} a_k(f)\overline{a_l(f)} r^{k+l} \delta_{kl} = 2\pi \sum_{k=0}^{+\infty} |a_k(f)|^2 r^{2k}\end{aligned}$$

Hence, $f \in \mathcal{H}^2(\mathbb{D})$ if and only if $(a_k(f))_{k\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{C})$.

(b) By similar calculations and arguments as above, for every $f, g \in \mathcal{H}^2(\mathbb{D})$ and for every $r \in (0,1)$ we have that

$$\begin{aligned}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})} \, dt &= \int_0^{2\pi} \sum_{k,l=0}^{+\infty} a_k(f)\overline{a_l(g)} r^{k+l} e^{i(k-l)t} \, dt \\ &= 2\pi \sum_{k,l=0}^{+\infty} a_k(f)\overline{a_l(g)} r^{k+l} \delta_{kl} = 2\pi \sum_{k=0}^{+\infty} a_k(f)\overline{a_l(g)} r^{2k}\end{aligned}$$

Since $(a_k(f))_{k\in\mathbb{N}}$, $(a_k(g))_{k\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{C})$ by (a), by Cauchy-Schwartz and dominated convergence we get that

$$\langle f, g \rangle = \lim_{r\to 1^{-1}} \sum_{k=0}^{+\infty} a_k(f)\overline{a_l(g)} r^{2k} = \sum_{k=0}^{+\infty} a_k(f)\overline{a_l(g)}.$$

(c) Exercises (a) and (b) above show that $(\mathcal{H}^2(\mathbb{D}), \langle \cdot, \cdot \rangle)$ is isometrically isomorphic to $\ell^2(\mathbb{N}, \mathbb{C})$. Hence, $(\mathcal{H}^2(\mathbb{D}), \langle \cdot, \cdot \rangle)$ is a Hilbert space. Moreover, since $(e^{(k)})_{k\in\mathbb{N}}$ given by $e^{(k)}_n := \delta_{nk}$ $(n, k \in \mathbb{N})$ is an orthonormal basis of $\ell^2(\mathbb{N}, \mathbb{C})$ which corresponds to $(\mathbb{D} \ni z \mapsto z^n \in \mathbb{C})_{n\in\mathbb{N}}$ through the given isometric isomorphism, the statement follows.

$\square$