Exercise 5.1 Let H be a Hilbert space, and let $A\colon H \to H$ be a bounded linear map.

(a) Let $y \in H$. Show that there exists a unique $z \in H$ so that $(Ax, y) = (x, z)$ for all $x \in H$.

(b) We define the *adjoint operator* A^* of A by setting $A^*y = z$ for $y \in H$ and z as in part (a). Show that $A^*\colon H \to H$ is a bounded linear operator.

(c) Prove that $\|A^*\|_{L(H)} = \|A\|_{L(H)}$.

(d) Show that $(\operatorname{ran}(A))^\perp = \ker(A^*)$.

Exercise 5.2 Let H be a real Hilbert space. Let $a\colon H \times H \to \mathbb{R}$ be bilinear and continuous; let $\Lambda \geq 0$ be such that $|a(x, y)| \leq \Lambda \|x\|_H \|y\|_H$ for all $x, y \in H$. Suppose that a is *coercive*, i.e. there exists $\lambda > 0$ so that $a(x, x) \geq \lambda \|x\|_H^2$ for all $x \in H$.

(a) Let $x \in H$. Show that there exists a unique vector $z \in H$ so that $a(x, y) = (z, y)$ for all $y \in H$.

(b) Define a map $A\colon H \to H$ by $x \mapsto z$, and show that A is linear and bounded with $\|A\|_{L(H)} \leq \Lambda$.

(c) Prove that A is injective.

Hint. Estimate (Ax, x) using the coercivity of a.

(d) Show that $\operatorname{ran}(A) = A(H)$ is closed.

(e) Show that A is surjective.

Hint. Notice that A^* is injective and use exercise 5.1-(d).

(f) Show that $A^{-1} \in L(H)$, and prove $\|A^{-1}\| \leq \lambda^{-1}$.

Exercise 5.3 The right-shift map $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ on the space $\ell^2(\mathbb{N})$ is given by

$$S((x_0, x_1, x_2, \dots)) := (0, x_0, x_1, \dots), \qquad \forall\, (x_0, x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

(a) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(b) Show that $S - \lambda I$ is invertible for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$.

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(c) Show that $S - \lambda I$ is injective for all $\lambda \in \mathbb{C}$. Show that the range of $S - \lambda I$ is not dense for $|\lambda| < 1$ whilst $S - \lambda I$ has dense range but it is not surjective for $|\lambda| = 1$.

Hint. For the dense range properties, use exercise 5.1-(d). For the failure of surjectivity, consider the sequence $\{y_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ given by $y_k := \lambda^{-k}(k+1)^{-1}$ for every $k \in \mathbb{N}$.

(d) Show that S has a left inverse in the sense that there exists an operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ such that $T \circ S = I$. Check that $S \circ T \neq I$.

Exercise 5.4 Define a map $T\colon C^0([0,1]) \to (L^1([0,1]))^*$ by

$$(Tu)(v) = \int_0^1 u(x)v(x)\,\mathrm{d}x \qquad \forall\, u \in C^0([0,1]),\ v \in L^1([0,1]).$$

(a) Show that T is continuous and injective.

(b) Show that $\|T\|_{L(C^0,(L^1)^*)} = 1$.

(c) Show that the range of T is closed, but not dense.