Exercise 5.1 Let H be a Hilbert space, and let $A\colon H \to H$ be a bounded linear map.

(a) Let $y \in H$. Show that there exists a unique $z \in H$ so that $(Ax, y) = (x, z)$ for all $x \in H$.

(b) We define the *adjoint operator* A^* of A by setting $A^*y = z$ for $y \in H$ and z as in part (a). Show that $A^*\colon H \to H$ is a bounded linear operator.

(c) Prove that $\|A^*\|_{L(H)} = \|A\|_{L(H)}$.

(d) Show that $(\operatorname{ran}(A))^\perp = \ker(A^*)$.

Solution.

(a) Let $y \in H$ and consider the linear functional $\lambda_y : H \to \mathbb{R}$ given by

$$\lambda_y(x) := (Ax, y), \qquad \forall\, x \in H.$$

Notice that $\lambda_y \in H^*$ with $\|\lambda_y\|_{H^*} \leq \|A\|_{L(H)}\|y\|_H$. Indeed, by Cauchy-Schwarz inequality and since A is bounded, we have

$$|\lambda_y(x)| = |(Ax, y)| \leq \|Ax\|_H\|y\|_H \leq (\|A\|_{L(H)}\|y\|_H)\|x\|_H, \qquad \forall\, x \in H.$$

Thus, by Riesz representation theorem there exists a unique $z \in H$ such that

$$(Ax, y) = \lambda_y(x) = (x, z), \qquad \forall\, x \in H.$$

(b) A straightforward computation shows that A^* is linear. Moreover,

$$\|A^*y\|_H = \|z\|_H = \sup_{\substack{x\in H \\ \|x\|_H\leq 1}} |(x, z)| = \sup_{\substack{x\in H \\ \|x\|_H\leq 1}} |\lambda_y(x)| = \|\lambda_y\|_{H^*} \leq \|A\|_{L(H)}\|y\|_H,$$

for every $y \in H$. Hence, A^* is bounded with $\|A^*\|_{L(H)} \leq \|A\|_{L(H)}$.

(c) We are just left to show that $\|A^*\|_{L(H)} \geq \|A\|_{L(H)}$. By definition and Cauchy-Schwarz inequality, for every $x \in H$ we have

$$\|Ax\|_H^2 = |(Ax, Ax)| = |(x, A^*(Ax))| \leq \|x\|_H\|A^*\|_{L(H)}\|Ax\|_H,$$

which implies

$$\|Ax\|_H = |(Ax, Ax)| = |(x, A^*(Ax))| \leq \|A^*\|_{L(H)}\|x\|_H.$$

The statement follows.

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(d) Let $x \in H$, Then $y \in (\operatorname{ran}(A))^\perp$ if and only if $(Ax, y) = (x, A^*y) = 0$ for every $x \in H$. On the other hand, $(x, A^*y) = 0$ for every $x \in H$ if and only if $y \in \ker(A^*)$. The statement follows.

□

Exercise 5.2 Let H be a real Hilbert space. Let $a\colon H \times H \to \mathbb{R}$ be bilinear and continuous; let $\Lambda \geq 0$ be such that $|a(x, y)| \leq \Lambda \|x\|_H \|y\|_H$ for all $x, y \in H$. Suppose that a is *coercive*, i.e. there exists $\lambda > 0$ so that $a(x, x) \geq \lambda \|x\|_H^2$ for all $x \in H$.

(a) Let $x \in H$. Show that there exists a unique vector $z \in H$ so that $a(x, y) = (z, y)$ for all $y \in H$.

(b) Define a map $A\colon H \to H$ by $x \mapsto z$, and show that A is linear and bounded with $\|A\|_{L(H)} \leq \Lambda$.

(c) Prove that A is injective.

Hint. Estimate (Ax, x) using the coercivity of a.

(d) Show that $\operatorname{ran}(A) = A(H)$ is closed.

(e) Show that A is surjective.

Hint. Notice that A^* is injective and use exercise 5.1-(d).

(f) Show that $A^{-1} \in L(H)$, and prove $\|A^{-1}\| \leq \lambda^{-1}$.

Solution.

(a) Fix any $x \in H$ and consider the functional $\lambda_x : H \to \mathbb{R}$ given by

$$\lambda_x(y) := a(x, y), \forall\, y \in H.$$

Notice that

$$|\lambda_x(y)| = |a(x, y)| \leq (\Lambda \|x\|_H)\|y\|_H, \qquad \forall\, y \in H.$$

This implies that $\lambda_x \in H^*$ with $\|\lambda_x\|_{H^*} \leq \Lambda \|x\|_H$. By Riesz representation theorem, there exists a unique $z \in H$ with $\|z\|_H = \|\lambda_x\|_{H^*} \leq \Lambda \|x\|_H$ such that

$$a(x, y) = \lambda_x(y) = (z, y) \qquad \forall\, y \in H.$$

(b) It is clear that

$$\|Ax\|_H = \|z\|_H \leq \Lambda\|x\|_H, \qquad \forall x \in H.$$

This implies that $A \in L(H)$ with $\|A\|_{L(H)} \leq \Lambda$.

(c) Notice that

$$(Ax, x) = (z, x) = a(x, x) \geq \lambda\|x\|_H^2, \qquad \forall\, x \in H.$$

Hence $Ax = 0$ implies $x = 0$ and the injectivity of A is proved.

(d) Notice that

$$\|Ax\|_H\|x\|_H \geq (Ax, x) \geq \lambda\|x\|_H^2, \qquad \forall\, x \in H,$$

which implies

$$\|Ax\|_H \geq \lambda\|x\|_H, \qquad \forall\, x \in H. \tag{1}$$

Now pick any sequence $\{Ax_k\}_{k\in\mathbb{N}} \subset \operatorname{ran}(A)$ such that $Ax_k \to y \in H$. Since $\{Ax_k\}_{k\in\mathbb{N}}$ si Cauchy, by linearity and the estimate (1) we conclude that $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy. Since H is complete, we have that there exists $X \in H$ such that $x_k \to x$. By the continuity of A and uniqueness of the limit, we have $Ax = y$ and we have proved that $\operatorname{ran}(A)$ is closed.

(e) Notice that A^* is injective. This follows because by estimate (1) it holds that

$$(x, A^*x) = (Ax, x) \geq \lambda\|x\|_H^2, \qquad \forall x \in H.$$

Then, by exercise 5.1-(d) we have $(\operatorname{ran}(A))^\perp = \ker(A^*) = \{0\}$ and we get $\overline{\operatorname{ran}(A)} = H$. But by the previous point we know that $\operatorname{ran}(A)$ is closed and the statement follows.

(f) Since $A \in L(H)$ is bijective and H is complete, we conclude that A has a continuous inverse $A^{-1} \in L(H)$. In order to estimate the norm of A^{-1}, we compute

$$\lambda\|A^{-1}x\|_H \leq \|A(A^{-1}x)\|_H = \|x\|_H, \qquad \forall\, x \in H.$$

This implies that $\|A^{-1}x\|_H \leq \lambda^{-1}\|x\|_H$ for every $x \in H$ and the statement follows.

□

Exercise 5.3 The right-shift map $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ on the space $\ell^2(\mathbb{N})$ is given by

$$S((x_0, x_1, x_2, \dots)) := (0, x_0, x_1, \dots), \qquad \forall\, (x_0, x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

(a) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(b) Show that $S - \lambda I$ is invertible for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$.

(c) Show that $S - \lambda I$ is injective for all $\lambda \in \mathbb{C}$. Show that the range of $S - \lambda I$ is not dense for $|\lambda| < 1$ whilst $S - \lambda I$ has dense range but it is not surjective for $|\lambda| = 1$.

Hint. For the dense range properties, use exercise 5.1-(d). For the failure of surjectivity, consider the sequence $\{y_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ given by $y_k := \lambda^{-k}(k+1)^{-1}$ for every $k \in \mathbb{N}$.

(d) Show that S has a left inverse in the sense that there exists an operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ such that $T \circ S = I$. Check that $S \circ T \neq I$.

Solution.

(a) We notice that

$$\|S((x_0, x_1, x_2, \dots))\|_{\ell^2}^2 = 0^2 + \sum_{n=0}^{+\infty} |x_k|^2 = \|(x_0, x_1, x_2, \dots)\|_{\ell^2}^2,$$

for every $(x_0, x_1, x_2, \dots) \in \ell^2(\mathbb{N})$. Hence, S is an isometry and the statement follows.

(b) Fix any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. We want to show that the equation

$$(y_0, y_1, y_2, \dots) = (S - \lambda I)((x_0, x_1, x_2, \dots)) = (-\lambda x_0, x_0 - \lambda x_1, x_1 - \lambda x_2, \dots) \quad (2)$$

has a unique solution $(x_0, x_1, x_2, \dots) \in \ell^2(\mathbb{N})$ for every given $(y_0, y_1, y_2, \dots) \in \ell^2(\mathbb{N})$. Equation (2) implies

$$\begin{cases} x_0 &= -\lambda^{-1} y_0 \\ x_1 &= \lambda^{-1}(x_0 - y_1) = \lambda^{-2} y_0 - \lambda^{-1} y_1 \\ x_2 &= \lambda^{-1}(x_1 - y_2) = \lambda^{-3} y_0 - \lambda^{-2} y_1 - \lambda^{-1} y_2 \\ &\vdots \end{cases}$$

Therefore, the unique sequence $(x_0, x_1, x_2, \dots)$ that solves equation (2) is given by

$$x_k := \sum_{j=0}^{k} \lambda^{-(k-j+1)} y_j = \lambda^{-1} \sum_{j=0}^{k} \lambda^{-(k-j)} y_j, \qquad \forall\, k \in \mathbb{N}. \quad (3)$$

We are just left to show that $(x_0, x_1, x_2, \dots) \in \ell^2$ (and this is where the hypothesis on λ is needed). Indeed, since $|\lambda| > 1$, we have that

$$\begin{aligned}\sum_{k=0}^{+\infty}|x_k|^2 &= \sum_{k=0}^{+\infty}\left|\lambda^{-1}\sum_{j=0}^{k}\lambda^{-(k-j)}y_j\right|^2 \le |\lambda|^{-2}\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{k}|\lambda^{-(k-j)}||y_j|\right)^2\\ &\le |\lambda|^{-2}\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{k}|\lambda^{-(k-j)}|^2\sum_{j=0}^{k}|y_j|^2\right) \le |\lambda|^{-2}\sum_{j=0}^{+\infty}|y_j|^2\sum_{k=0}^{+\infty}\sum_{j=0}^{k}|\lambda^{-(k-j)}|^2\\ &= |\lambda|^{-2}\|(y_0, y_1, y_2, \dots)\|_{\ell^2}\sum_{k=0}^{+\infty}(k+1)|\lambda|^{-2k} < +\infty.\end{aligned}$$

(c) First, we show that $S - \lambda I$ is injective for every $\lambda \in \mathbb{C}$. If $\lambda = 0$, then clearly $Sx = 0$ implies $x = 0$ (recall that S is an isometry) and the statement is proved. If $\lambda \neq 0$, the same computation that we have made in point (b) leads to conclude that $(S - \lambda I)x = 0$ forces $x = 0$ for general $\lambda \in \mathbb{C} \setminus \{0\}$ and we are done.

Second, we notice that $\overline{\operatorname{ran}(S - \lambda I)} = \ell^2(\mathbb{N})$ if and only if $(\operatorname{ran}(S - \lambda I))^\perp = \{0\}$. By exercise 5.1-(d), we have $(\operatorname{ran}(S - \lambda I))^\perp = \ker(S^* - \overline{\lambda} I)$. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be given by

$$T((x_0, x_1, x_2, \dots)) := (x_1, x_2, x_3, \dots), \qquad \forall\,(x_0, x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

Notice that $S^* = T$. Hence, we are left to study $\ker(T - \overline{\lambda} I)$ to understand if $S - \lambda I$ has dense range. We want to solve the equation $(T - \overline{\lambda} I)((x_0, x_1, x_2, \dots)) = 0$, which leads to

$$x_k := \overline{\lambda}^k x_0, \qquad \forall k \in \mathbb{N}.$$

Such solutions belong to $\ell^2(\mathbb{N})$ if and only if $|\overline{\lambda}| = |\lambda| < 1$, in which case we obtain $\ker(L - \overline{\lambda} I) = \operatorname{span}_{\mathbb{C}}\{(1, \overline{\lambda}, \overline{\lambda}^2, \dots)\} \neq \{0\}$. Otherwise, we have $\ker(L - \overline{\lambda} I) = \{0\}$. We conclude that if $|\lambda| < 1$ then the range of $S - \lambda I$ is not dense in $\ell^2(\mathbb{N})$ whilst for $|\lambda| = 1$ we get that $S - \lambda I$ has dense range.

In order to prove that even in case $|\lambda| = 1$ we have $\operatorname{ran}(S - \lambda I) \neq \ell^2(\mathbb{N})$, consider the sequence $\{y_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ given by $y_k := \lambda^{-k}(k+1)^{-1}$ for every $k \in \mathbb{N}$. Then, if we try to solve the equation $(S - \lambda I)((x_0, x_1, x_2, \dots)) = (y_0, y_1, y_2, \dots)$ we get

$$\begin{cases} x_0 &= -\lambda^{-1}\\ x_1 &= \lambda^{-1}(x_0 - y_1) = -\lambda^{-2}(1 + 2^{-1})\\ x_2 &= \lambda^{-1}(x_1 - y_2) = -\lambda^{-3}(1 + 2^{-1} + 3^{-1})\\ &\vdots \end{cases}$$

But the solution given by the previous equation does not belong to ℓ^2 because

$$\sum_{k=0}^{+\infty} |x_k|^2 = \sum_{k=0}^{+\infty} \left| \sum_{j=0}^{k} \frac{1}{j+1} \right|^2 \geq \sum_{k=0}^{+\infty} 1 = +\infty.$$

The statement follows.

(d) It is straightforward to check that $T \circ S = I$. Nevertheless, we have that

$$(S \circ T)((x_0, x_1, x_2, \dots)) = S((x_1, x_2, x_3, \dots)) = (0, x_1, x_2, \dots) \neq (x_0, x_1, x_2, \dots)$$

for every $(x_0, x_1, x_2, \dots) \in \ell^2(\mathbb{N})$ such that $x_0 \neq 0$. The statement follows.

□

Exercise 5.4 Define a map $T\colon C^0([0,1]) \to (L^1([0,1]))^*$ by

$$(Tu)(v) = \int_0^1 u(x)v(x)\,\mathrm{d}x \qquad \forall\, u \in C^0([0,1]),\ v \in L^1([0,1]).$$

(a) Show that T is continuous and injective.

(b) Show that $\|T\|_{L(C^0,(L^1)^*)} = 1$.

(c) Show that the range of T is closed, but not dense.

Solution.

(a) We estimate

$$|(Tu)(v)| \leq \int_0^1 |u(x)||v(x)|\,dx \leq \|u\|_{C^0} \|v\|_{L^1},$$

for every $u \in C^0([0,1])$, $v \in L^1([0,1])$. It follows that $\|Tu\|_{(L^1)^*} \leq \|u\|_{C^0}$, for every $u \in C^0([0,1])$ and this suffices to prove that T is continuous.

To show injectivity, we assume that $u \in C^0([0,1])$ and we assume that $Tu = 0$. Since $u \in L^1([0,1])$, we have that

$$0 = (Tu)(u) = \int_0^1 |u(x)|^2\,dx,$$

which implies $u = 0$ a.e. on $[0,1]$. By continuity of u we get $u = 0$ on th whole interval $[0,1]$.

(b) Again, notice that $u \in L^1([0,1])$ and $\|u\|_{L^1} \leq \|u\|_{C^0}$, for every $u \in C^0([0,1])$. By Hölder inequality, we have that

$$\|Tu\|_{(L^1)^*} \frac{\|u\|_{L^1}}{\|u\|_{C^0}} \geq |(Tu)(u)| = \int_0^1 |u(x)|^2 \, dx \geq \|u\|_{L^1}$$

which implies $\|Tu\|_{(L^1)^*} \geq \|u\|_{C^0}$, for every $u \in C^0([0,1])$. The statement follows. Nevertheless, by point (a), we know that $\|Tu\|_{(L^1)^*} \leq \|u\|_{C^0}$, for every $u \in C^0([0,1])$. Hence, we obtain that $\|Tu\|_{(L^1)^*} = \|u\|_{C^0}$, for every $u \in C^0([0,1])$ and the statement follows.

(c) First, we show that $\operatorname{ran}(T)$ is closed. Pick any sequence $\{Tu_k\}_{k\in\mathbb{N}} \subset \operatorname{ran}(T)$ such that $Tu_k \to \lambda$ in $(L^1([0,1]))^*$. Then, since T is an isometry, we have

$$\|u_k - u_h\|_{C^0} = \|T(u_k - u_h)\|_{(L^1)^*} = \|Tu_k - Tu_h\|_{(L^1)^*} \to 0^+$$

as $k, h \to +\infty$. We conclude that $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,1])$. Since $C^0([0,1])$ is complete, there exists $u \in C^0([0,1])$ such that $u_k \to u$ in $C^0([0,1])$. We claim that $Tu = \lambda$. Indeed, by uniform convergence of the u_k to u, we get

$$\lambda(v) = \lim_{k\to+\infty} (Tu_k)(v) = \lim_{k\to+\infty} \int_0^1 u_k(x)v(x) \, dx = \int_0^1 u(x)v(x) \, dx = (Tu)(v),$$

for every $v \in L^1([0,1])$. Hence, $\lambda = Tu \in \operatorname{ran}(T)$ and the closure of $\operatorname{ran}(T)$ is proved.

Now assume by contradiction that $\operatorname{ran}(T)$ is not dense in $(L^1([0,1]))^*$. Then, since $\operatorname{ran}(T)$ is closed, we have $\operatorname{ran}(T) = (L^1([0,1]))^*$. Consider the functional $\xi : L^1([0,1]) \to \mathbb{R}$ given by

$$\xi(v) := \int_0^{\frac{1}{2}} v(x) \, dx, \qquad \forall\, v \in L^1([0,1]).$$

It is straightforward that $\xi \in (L^1([0,1]))^*$. We claim that $\xi \notin \operatorname{ran}(T)$, which would produce a contradiction. Indeed, assume that there exists $u \in C^0([0,1])$ such that

$$\int_0^{\frac{1}{2}} v(x) \, dx = \xi(v) = (Tu)(v) = \int_0^1 u(x)v(x) \, dx, \qquad \forall\, v \in L^1([0,1]).$$

This implies

$$\int_0^1 (u(x) - \chi_{[0,\frac{1}{2}]})v(x) \, dx = 0, \qquad \forall\, v \in L^1([0,1]),$$

where $\chi_{[0,\frac{1}{2}]}$ denotes the indicator function of the interval $[0, \frac{1}{2}]$. By the fundamental lemma of calculus of variations this implies that $u = \chi_{[0,\frac{1}{2}}$ a.e. on $[0,1]$ and this contradicts the continuity of u.

□