Functional Analysis I
Exercise Sheet 6

Exercise 6.1 Let H be a separable infinite-dimensional Hilbert space with complete orthonormal basis $(e_i)_{i\in\mathbb{N}}$. Suppose $A\colon H \to H$ is a bounded linear map with the property that

$$\|A\|_{HS}^2 := \sum_{i=1}^{\infty} \|Ae_i\|_H^2 < \infty.$$

Operators A with this property are called *Hilbert–Schmidt operators*, and $\|A\|_{HS}$ is their so-called *Hilbert–Schmidt norm.*

(a) Prove that $\|A\|_{HS}$ is independent of the choice of the complete orthonormal basis.

(b) Show that $\|A\|_{L(H)} \leq \|A\|_{HS}$.

(c) Find a bounded operator that is not Hilbert–Schmidt.

Exercise 6.2 Specifying Parseval's identity for the Fourier transform to $f(x) = x$ (seen as an element of $L^2([0, 2\pi])$), show that

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Exercise 6.3 Let $(H, (\,\cdot\,,\,\cdot\,)_H)$ be a real, infinite-dimensional Hilbert space. Let $x \in H$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.

(a) Prove that the weak convergence $x_n \xrightarrow{w} x$ in H and the convergence of the norms $\|x_n\|_H \to \|x\|_H$ in $\mathbb{R}$ implies (strong) convergence $x_n \to x$ in H, i.e. $\|x_n - x\|_H \to 0$.

(b) Suppose $x_n \xrightarrow{w} x$ and $\|y_n - y\|_H \to 0$, where $(y_n)_{n\in\mathbb{N}}$ is another sequence in H and $y \in H$. Prove that $(x_n, y_n)_H \to (x, y)_H$.

(c) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of $(H, (\,\cdot\,,\,\cdot\,)_H)$. Prove that $e_n \xrightarrow{w} 0$.

(d) Given any $x \in H$ with $\|x\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x$.

(e) Let the functions $f_n : [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for all $\mathbb{N} \setminus \{0\}$. Prove that $f_n \xrightarrow{w} 0$ in $L^2([0, 2\pi])$.

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Exercise 6.4

(a) Show that $L^\infty([0,1])$ is not separable.

Hint. Consider for $x_0 \in [0,1]$ the step function $f_{x_0} \in L^\infty([0,1])$, defined by $f_{x_0}(x) = 1$ for $x \leq x_0$ and $f_{x_0}(x) = 0$ for $x > x_0$.

(b) For any $1 \leq p \leq +\infty$, find an explicit sequence in $(L^p([0,1]), \|\cdot\|_{L^p})$ which is bounded but does not have a convergent subsequence.