Exercise 6.1 Let H be a separable infinite-dimensional Hilbert space with complete orthonormal basis $(e_i)_{i\in\mathbb{N}}$. Suppose $A\colon H \to H$ is a bounded linear map with the property that

$$\|A\|_{HS}^2 := \sum_{i=1}^{\infty} \|Ae_i\|_H^2 < \infty.$$

Operators A with this property are called *Hilbert–Schmidt operators*, and $\|A\|_{HS}$ is their so-called *Hilbert–Schmidt norm.*

(a) Prove that $\|A\|_{HS}$ is independent of the choice of the complete orthonormal basis.

(b) Show that $\|A\|_{L(H)} \leq \|A\|_{HS}$.

(c) Find a bounded operator that is not Hilbert–Schmidt.

Solution.

(a) Notice that since A is bounded, by Exercise 5.1 A has a well-defined, bounded adjoint operator A^*. Let$(e_i)_{i\in\mathbb{N}}$ and $(\tilde{e}_i)_{i\in\mathbb{N}}$ be two (possibly coinciding) complete orthonormal basis of H. Then, by Parseval's identity we have

$$\sum_{i=0}^{+\infty}\|A\tilde{e}_i\|_H^2 = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}|(A\tilde{e}_i, e_j)_H|^2 = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}|(\tilde{e}_i, A^*e_j)_H|^2 = \sum_{j=0}^{+\infty}\|A^*e_j\|_H^2. \tag{1}$$

By exploiting (1) with $\tilde{e}_i = e_i$ for every $i \in \mathbb{N}$ we get

$$\sum_{i=0}^{+\infty}\|Ae_i\|_H^2 = \sum_{i=0}^{+\infty}\|A^*e_i\|_H^2.$$

The statement follows.

(b) Given any $x \in H$ such that $\|x\|_H = 1$ we extend x to a complete orthonormal basis $\{x, e_1, e_2, \dots\}$ of H. Then, by definition and point (a), we get

$$\|Ax\|_H^2 \leq \|Ax\|_H^2 + \sum_{i=1}^{+\infty}\|Ae_i\|_H^2 = \|A\|_{HS}^2.$$

Hence, we have $\|Ax\|_H \leq \|A\|_{HS}$, for every $x \in H$ such that $\|x\|_H = 1$. By taking the supremum over x in the previous inequality, the statement follows.

(c) The identity operator does the job.

□

Last modified: 4 November 2022

Exercise 6.2 Specifying Parseval's identity for the Fourier transform to $f(x) = x$ (seen as an element of $L^2([0, 2\pi])$), show that

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Solution. Recall that a Hilbertian basis of $L^2([0, 2\pi])$ is given by

$$\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos(kx)\right\}_{k\geq 1} \cup \left\{\frac{1}{\sqrt{\pi}}\sin(kx)\right\}_{k\geq 1}.$$

Hence, we have

$$\frac{(2\pi)^3}{3} = \|f\|^2_{L^2([0,2\pi])} = |a_0|^2 + \sum_{k=1}^{+\infty} |a_k|^2 + |b_k|^2$$

with

$$\begin{aligned} a_0 &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x\,dx = \frac{(2\pi)^{\frac{3}{2}}}{2}, \\ a_k &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} x\cos(kx)\,dx = 0 \qquad (k \geq 1), \\ b_k &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} x\sin(kx)\,dx = -\frac{2\sqrt{\pi}}{k} \qquad (k \geq 1). \end{aligned}$$

Hence,

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{1}{4\pi}\left(\frac{(2\pi)^3}{3} - \frac{(2\pi)^3}{4}\right) = \frac{\pi^2}{6}.$$

□

Exercise 6.3 Let $(H, (\,\cdot\,,\cdot\,)_H)$ be a real, infinite-dimensional Hilbert space. Let $x \in H$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.

(a) Prove that the weak convergence $x_n \xrightarrow{w} x$ in H and the convergence of the norms $\|x_n\|_H \to \|x\|_H$ in $\mathbb{R}$ implies (strong) convergence $x_n \to x$ in H, i.e. $\|x_n - x\|_H \to 0$.

(b) Suppose $x_n \xrightarrow{w} x$ and $\|y_n - y\|_H \to 0$, where $(y_n)_{n\in\mathbb{N}}$ is another sequence in H and $y \in H$. Prove that $(x_n, y_n)_H \to (x, y)_H$.

(c) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of $(H, (\,\cdot\,,\cdot\,)_H)$. Prove that $e_n \xrightarrow{w} 0$.

(d) Given any $x \in H$ with $\|x\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x$.

(e) Let the functions $f_n : [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for all $\mathbb{N} \setminus \{0\}$. Prove that $f_n \xrightarrow{w} 0$ in $L^2([0, 2\pi])$.

Solution.

(a) We compute

$$\|x_n - x\|_H^2 = (x_n - x, x_n - x)_H = \|x_n\|^2 + \|x\|_H^2 - 2(x_n, x)_H. \tag{2}$$

We notice that

$$|(x_n, x) - \|x\|_H^2| = |(x_n - x, x)_H| \to 0 \quad (n \to +\infty)$$

because $x_n \xrightarrow{w} x$ in H. Hence, by passing to the limit as $n \to +\infty$ in (2) the statement follows.

(b) Since $x_n \xrightarrow{w} x$, we have that $(x_n)_{n\in\mathbb{N}}$ is bounded, i.e. there exists $C > 0$ such that $\|x_n\|_H \leq C$ for every $n \in \mathbb{N}$. We estimate

$$\begin{aligned}
|(x_n, y_n)_H - (x, y)_H| &= |(x_n, y_n - y)_H + (x_n - x, y)_H| \\
&\leq |(x_n, y_n - y)_H| + |(x_n - x, y)_H| \\
&\leq \|x_n\|_H \|y_n - y\|_H + |(x_n - x, y)_H| \\
&\leq C\|y_n - y\|_H + |(x_n - x, y)_H|.
\end{aligned}$$

By weak convergence of $(x_n)_{n\in\mathbb{N}}$ to x and strong convergence of $(y_n)_{n\in\mathbb{N}}$ to y the statement follows.

(c) Notice that, by Bessel's inequality the series

$$\sum_{n=0}^{+\infty} |(x, e_n)_H|^2 \leq \|x\|_H^2 < +\infty$$

is convergent for every $x \in H$. Then, we conclude that $|(x, e_n)_H|^2 \to 0$ as $n \to +\infty$ for every $x \in H$. The statement follows.

(d) Let $x \in H$ satisfy $\|x\|_H \leq 1$. If $x = 0$, by point (c) we have that any orthonormal system does the job. If $x \neq 0$, then an orthonormal system $(e_n)_{n\in\mathbb{N}}$ of H with $e_0 := \|x\|^{-1}x$ can be constructed via the Gram-Schmidt algorithm. For $n \in \mathbb{N}$, define

$$x_n := x + \left(\sqrt{1 - \|x\|_H^2}\right) e_{n+1}.$$

Then, since $x \perp e_{n+1}$ for every $n \in \mathbb{N}$ we have $\|x_n\|_H^2 = \|x\|_H^2 + 1 - \|x\|_H^2 = 1$, for every $n \in \mathbb{N}$. Moreover, $x_n \xrightarrow{w} x$ follows by $e_{n+1} \xrightarrow{w} 0$.

(e) Given $f_n : [0, 2\pi] \to \mathbb{R}$ as in the statement, we have that $\left(\frac{1}{\sqrt{\pi}} f_n\right)_{n \in \mathbb{N} \setminus \{0\}}$ is an orthonormal system of $L^2([0, 2\pi])$, because

$$\begin{aligned}\int_0^{2\pi} \sin(mt)\sin(nt)\,dt &= \frac{1}{2}\int_0^{2\pi} \Big(\cos((m-n)t) - \cos((m+n)t)\Big)\,dt \\ &= \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}\end{aligned}$$

for every $m, n \in \mathbb{N} \setminus \{0\}$. Hence, by point (c) the statement follows.

□

Exercise 6.4

(a) Show that $L^\infty([0,1])$ is not separable.

Hint. Consider for $x_0 \in [0,1]$ the step function $f_{x_0} \in L^\infty([0,1])$, defined by $f_{x_0}(x) = 1$ for $x \leq x_0$ and $f_{x_0}(x) = 0$ for $x > x_0$.

(b) For any $1 \leq p \leq +\infty$, find an explicit sequence in $(L^p([0,1]), \|\cdot\|_{L^p})$ which is bounded but does not have a convergent subsequence.

Solution.

(a) Consider for $x_0 \in [0,1]$ the step function $f_{x_0} \in L^\infty([0,1])$, defined by $f_{x_0}(x) = 1$ for $x \leq x_0$ and $f_{x_0}(x) = 0$ for $x > x_0$. Consider the following family of open balls in $L^\infty([0,1])$:

$$\mathscr{B} := \Big\{ B_{\frac{1}{2}}(f_{x_0}) \text{ for every } x_0 \in [0,1] \Big\},$$

where $B_\rho(f_0) := \{f \in L^\infty([0,1]) \text{ s.t. } \|f - f_0\|_{L^\infty([0,1])} < \rho\}$ for every $f_0 \in L^\infty([0,1])$ and $\rho > 0$.

Clearly $\mathscr{B}$ is uncountable. Moreover, $\mathscr{B}$ is made of disjoint balls because

$$\|f_{x_1} - f_{x_2}\|_{L^\infty([0,1])} = 1, \qquad \forall\, x_1, x_2 \in [0,1] \text{ s.t. } x_1 \neq x_2.$$

Let S be any dense set in $L^\infty([0,1])$. Clearly S must intersect all of the open balls in $\mathscr{B}$ by definition. This implies that S is uncountable and the statement follows.

(b) For every $n \in \mathbb{N}$ we divide $[0,1]$ into 2^n sub intervals $I_1, ..., I_{2^n}$ of equal length and we consider the function $f_n : [0,1] \to \mathbb{R}$ on each I_k to be $-\frac{1}{2}$ if k is odd and $\frac{1}{2}$ is k is even. More precisely,

$$f_n(x) := \begin{cases} -\frac{1}{2} & \text{if } \exists\, k \in \mathbb{N} : \, 2^n x \in [2k-2, 2k-1) \\ \frac{1}{2} & \text{else.} \end{cases}$$

By construction, $\|f_n\|_{L^p([0,1])} = \frac{1}{2}$, for every $n \in \mathbb{N}$ and for every $1 \leq p \leq +\infty$. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p([0,1])$, for every $1 \leq p \leq +\infty$. Nevertheless, for every $n, m \in \mathbb{N}$ with $n \neq m$ we have

$$\begin{aligned} \|f_n - f_m\|^p_{L^p([0,1])} &= \frac{1}{2} \qquad \forall\, 1 \leq p < +\infty, \\ \|f_n - f_m\|_{L^\infty([0,1])} &= \frac{1}{2}. \end{aligned}$$

Consequently, $(f_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence.

□