Exercise 7.1 Let $(X, \|\cdot\|_X)$ be a normed space of finite dimension $d < +\infty$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that weak convergence $x_n \xrightarrow{\text{w}} x$ for $n \to +\infty$ implies $\|x_n - x\|_X \to 0$ for $n \to +\infty$.

Exercise 7.2

(a) Show that the norm-closed unit ball of c_0 is not weakly sequentially compact; recall that $(c_0)^* \cong \ell^1$ (see Exercise 2.1-(a)).

Hint. Consider the sequence

$$\begin{aligned} x_0 &:= (1, 0, 0, 0, \dots) \\ x_1 &:= (1, 1, 0, 0, \dots) \\ x_2 &:= (1, 1, 1, 0, \dots) \\ &\vdots \end{aligned}$$

(b) Show that the unit ball in c_0 is also not weakly compact.

Hint. Consider the sets $A_k := \{x_k, x_{k+1}, x_{k+2}, \dots\}$ for $k \in \mathbb{N}$ and recall that a topological space is compact if and only if every collection of closed sets having the finite intersection property (i.e. the intersection of an arbitrary finite number of its elements is non-empty) has non-empty intersection.

Exercise 7.3 Let X be a real vector space.

(a) Let $n \in \mathbb{N}$ and let $\varphi_1, \varphi_2, \dots, \varphi_n, \psi : X \to \mathbb{R}$ be linear functionals. Prove that the following are equivalent:

(i) There exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ satisfying $\psi = \sum_{k=1}^{n} \lambda_k \varphi_k$.

(ii) There is a constant $C \in (0, +\infty)$ such that $|\psi(x)| \leq C \max_{1\leq k\leq n} |\varphi_k(x)|$ for all $x \in X$.

(iii) $\ker(\psi) \supset \bigcap_{k=1}^{n} \ker(\varphi_k)$.

(b) Let $F \subset \{f : X \to \mathbb{R} : f \text{ is linear}\}$ be a family of linear functionals and let $\mathcal{U}_F$ be the topology on X induced by F, i.e. the coarsest topology on X such that each element of F is continuous from $(X, \mathcal{U}_F)$ onto $\mathbb{R}$ with the standard euclidean topology. Prove that

$$\operatorname{span}(F) = \{\varphi : X \to \mathbb{R} : \varphi \text{ is } \mathcal{U}_F\text{-continuous and linear}\}.$$

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(c) Suppose X is a normed space. Consider a weak*-continuous linear functional $\varphi : X^* \to \mathbb{R}$. Prove that there is $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in X^*$.

Exercise 7.4 Let $(X, \|\cdot\|_X)$ be a normed space and let τ_w denote the weak topology on X. This exercise's goal is to show that τ_w is not metrizable if X is infinite-dimensional. Let us start by recalling what a neighbourhood basis is and what it means for a topology to be metrizable:

- (*Neighbourhood basis*) Let (Y, τ) be a topological space. Denoting the set of all neighbourhoods of a point $y \in Y$ by
$$\mathcal{U}_y = \{U \subset Y : \exists O \in \tau \text{ s.t. } y \in O \subset U\},$$
we call $\mathcal{B}_y \subset \mathcal{U}_y$ as *neighbourhood basis* of y in (Y, τ) if $\forall U \in \mathcal{U}_y \, \exists V \in \mathcal{B}_y$ s.t. $V \subset U$.
- (*Metrizability*) A topological space (Y, τ) is called *metrizable* if there exists a metric $d : Y \times Y \to \mathbb{R}$ on Y such that, denoting $B_\varepsilon(a) = \{y \in Y : d(y, a) < \varepsilon\}$ (for $a \in Y$, $\varepsilon \in (0, +\infty)$), there holds
$$\tau = \{O \subset Y \text{ s.t. } \forall a \in O \, \exists \varepsilon > 0 : B_\varepsilon(a) \subset O)\}.$$

(a) Show that any metrizable topology τ satisfies the first axiom of countability, which means that each point has a countable neighbourhood basis.

(b) Prove that
$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}(-\varepsilon, \varepsilon) \text{ s.t. } n \in \mathbb{N},\, f_1, f_2, \dots, f_n \in X^*,\, \varepsilon > 0 \right\}$$
is a neighbourhood basis of $0 \in X$ in (X, τ_w).

(c) Show that if (X, τ_w) is first countable, then $(X^*, \|\cdot|_{X^*})$ admits a countable algebraic basis.

Hint. Suppose (X, τ_w) is first countable and let $(U_j)_{j \in \mathbb{N}}$, be a countable neighborhood basis of 0. By part (b), for each U_j there exists an element
$$B_j = \bigcap_{k=1}^{n_j} f_{j,k}^{-1}((-\varepsilon_j, \varepsilon_j)) \in \mathcal{B}$$
contained in U_j, for some $n_j \in \mathbb{N}$, $f_{j,k} \in X^*$, $\varepsilon_j > 0$. Let now $\lambda \in X^*$, and consider the weakly open set $\lambda^{-1}((-1, 1))$. This contains B_j for some j. Show using Exercise 7.3-(a) that λ is a linear combination of $f_{j,k}$, $k = 1, \dots, n_j$. Conclude that X^* admits a countable algebraic basis.

(d) Assume that X is infinite-dimensional and conclude from (a), (c) and Exercise 3.2-(b) that (X, τ_w) is not metrizable.