Exercise 7.1 Let $(X, \|\cdot\|_X)$ be a normed space of finite dimension $d < +\infty$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that weak convergence $x_n \xrightarrow{\text{w}} x$ for $n \to +\infty$ implies $\|x_n - x\|_X \to 0$ for $n \to +\infty$.

Solution. Let $\{e_1, \ldots, e_d\}$ be a basis for the finite dimensional normed space $(X, \|\cdot\|_X)$. Then, every element $x \in X$ is of the form

$$x = \sum_{k=1}^{d} x^k e_k$$

for uniquely determined $x^1, \ldots, x^d$ (the superscripts are upper indices, not exponents). For $k \in \{1, \ldots, d\}$, we consider the linear maps $e_k^* : X \to \mathbb{R}$ given by $e_k^*(x) := x^k$. In fact, $e_k^* \in X^*$ since $|e_k^*(x)| = |x^k| \leq \|x\|_1$, where

$$\|x\|_1 := \sum_{k=1}^{d} |x^k|$$

defines a norm on X which must be equivalent to $\|\cdot\|_X$ since X is finite dimensional.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \xrightarrow{\text{w}} x$ for some $x \in X$ as $n \to +\infty$, then for every $k \in \{1, \ldots, d\}$ we have

$$\lim_{n\to+\infty} x_n^k = \lim_{n\to+\infty} e_k^*(x_n) = e_k^*(x) = x^k.$$

This implies $\|x_n - x\|_1 \to 0$ and by equivalence of norms $\|x_n - x\|_X \to 0$ as $n \to +\infty$. □

Exercise 7.2

(a) Show that the norm-closed unit ball of c_0 is not weakly sequentially compact; recall that $(c_0)^* \cong \ell^1$ (see Exercise 2.1-(a)).

Hint. Consider the sequence

$$\begin{aligned} x_0 &:= (1, 0, 0, 0, \ldots) \\ x_1 &:= (1, 1, 0, 0, \ldots) \\ x_2 &:= (1, 1, 1, 0, \ldots) \\ &\vdots \end{aligned}$$

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(b) Show that the unit ball in c_0 is also not weakly compact.

Hint. Consider the sets $A_k := \{x_k, x_{k+1}, x_{k+2}, \dots\}$ for $k \in \mathbb{N}$ and recall that a topological space is compact if and only if every collection of closed sets having the finite intersection property (i.e. the intersection of an arbitrary finite number of its elements is non-empty) has non-empty intersection.

Solution.

(a) Fix any $y = (y^0, y^1, y^2, \dots) \in \ell^1$ and compute

$$|\langle x_k - x, y\rangle| \le \sum_{n=0}^{+\infty} |x_k^n - x^n||y^n| = \sum_{n=k+1}^{+\infty} |y^n| \to 0^+ \qquad (k \to +\infty).$$

Now assume by contradiction that there exists a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ such that $x_{k_j} \xrightarrow{w} z$ for some $z \in c_0$. But then, since $(c_0)^* \cong \ell^1$, we have

$$\langle z - x, y\rangle = \lim_{j\to+\infty} \langle z - x_{k_j}, y\rangle + \lim_{j\to+\infty} \langle x_{k_j} - x, y\rangle = 0, \qquad \forall\, y \in \ell^1.$$

Again, since $(c_0)^* \cong \ell^1$ this implies $z = x = (1, 1, 1, \dots) \notin c_0$ which is a contradiction. The statement follows.

(b) By point (a), the sets A_k have no accumulation point with respect to the weak topology. Hence, they are all weakly closed. Moreover, any finite intersection of the A_k is non-empty whilst the intersections of all of them is empty. Hence, the statement follows.

□

Exercise 7.3 Let X be a real vector space.

(a) Let $n \in \mathbb{N}$ and let $\varphi_1, \varphi_2, \dots, \varphi_n, \psi : X \to \mathbb{R}$ be linear functionals. Prove that the following are equivalent:

(i) There exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ satisfying $\psi = \sum_{k=1}^{n} \lambda_k \varphi_k$.

(ii) There is a constant $C \in (0, +\infty)$ such that $|\psi(x)| \le C \max_{1\le k\le n} |\varphi_k(x)|$ for all $x \in X$.

(iii) $\ker(\psi) \supset \bigcap_{k=1}^{n} \ker(\varphi_k)$.

(b) Let $F \subset \{f : X \to \mathbb{R} : f \text{ is linear}\}$ be a family of linear functionals and let $\mathcal{U}_F$ be the topology on X induced by F, i.e. the coarsest topology on X such that each element of F is continuous from $(X, \mathcal{U}_F)$ onto $\mathbb{R}$ with the standard euclidean topology. Prove that

$$\operatorname{span}(F) = \{\varphi : X \to \mathbb{R} : \varphi \text{ is } \mathcal{U}_F\text{-continuous and linear}\}.$$

(c) Suppose X is a normed space. Consider a weak*-continuous linear functional $\varphi : X^* \to \mathbb{R}$. Prove that there is $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in X^*$.

Solution.

(a) First we show that (i) $\Rightarrow$ (ii). With $\lambda_1, \dots, \lambda_n$ such that $\psi = \sum_{k=1}^{n} \lambda_k \varphi_k$ we obtain

$$|\psi(x)| \leq \sum_{k=1}^{n} \lambda_k |\varphi_k(x)| \leq \left(\sum_{k=1}^{n} |\lambda_k| \right) \max_{1 \leq k \leq n} |\varphi_k(x)|.$$

Hence, (ii) holds with $C := \sum_{k=1}^{n} |\lambda_k|$.

Since (ii) $\Rightarrow$ (iii) is trivial, we are juts left to show that (iii) $\Rightarrow$ (i). Consider the linear function $\phi : X \to \mathbb{R}^n$ given by

$$\phi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)) \qquad \text{for all } x \in X.$$

Notice that for all $x, y \in X$ it holds that $x - y \in \ker(\phi)$ if and only if $\varphi_k(x) = \varphi_k(y)$ for all $k \in \{1, 2, \dots, n\}$. Thus, $X/\ker(\phi)$ is isomorphic to $\operatorname{ran}(\phi)$. Moreover, since $\ker(\psi) \supseteq \bigcap_{k=1}^{n} \ker(\varphi_k) = \ker(\phi)$, we see that there is a well-defined linear map $l : \operatorname{ran}(\phi) \to \mathbb{R}$ satisfying

$$l(\phi(x)) = \psi(x) \qquad \text{for all } x \in X.$$

We know from linear algebra (if you insist you can also invoke the Hahn-Banach theorem) that there exists a linear extension $L : \mathbb{R}^n \to \mathbb{R}$ of l. Moreover we know from linear algebra (if you insist you can also invoke Riesz's representation theorem for Hilbert spaces) that there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ such that $L(y) = \sum_{k=1}^{n} \lambda_k y_k$ for all $y = (y_1, y_2 \dots, y_n) \in \mathbb{R}^n$. This ensures in particular that

$$\psi(x) = l(\phi(x)) = L(\phi(x)) = \sum_{k=1}^{n} \lambda_k \varphi_k(x) \qquad \text{for all } x \in X.$$

(b) If $\varphi : X \to \mathbb{R}$ is linear and $\mathcal{U}_F$-continuous, then the set $\varphi^{-1}((-1,1))$ is $\mathcal{U}_F$-open. Hence, there are $f_1, \ldots, f_n \in F$ and $\varepsilon \in (0,\infty)$ such that

$$\varphi^{-1}((-1,1)) \supseteq \bigcap_{k=1}^{n} f_k^{-1}((-\varepsilon,\varepsilon)).$$

By linearity, we infer for every $m \in \mathbb{N}$ that

$$\varphi^{-1}\left(\left(-\frac{1}{m},\frac{1}{m}\right)\right) \supseteq \bigcap_{k=1}^{n} f_k^{-1}\left(\left(-\frac{\varepsilon}{m},\frac{\varepsilon}{m}\right)\right).$$

Letting $m \to \infty$, we obtain that $\varphi(x) = 0$ for all $x \in \bigcap_{k=1}^{m} \ker(f_k)$. Part (a) above now ensures that $\varphi \in \operatorname{span}(\{f_k \mid k \in \{1,2,\ldots,n\}\}) \subseteq \operatorname{span}(F)$.

(c) This follows immediately from (b) when taking $F = \{X^* \ni \varphi \mapsto \varphi(x) \in \mathbb{R} : x \in X\}$ and noticing that $F = \operatorname{span}(F)$.

□

Exercise 7.4 Let $(X, \|\cdot\|_X)$ be a normed space and let τ_{w} denote the weak topology on X. This exercise's goal is to show that τ_{w} is not metrizable if X is infinite-dimensional. Let us start by recalling what a neighbourhood basis is and what it means for a topology to be metrizable:

- (*Neighbourhood basis*) Let (Y,τ) be a topological space. Denoting the set of all neighbourhoods of a point $y \in Y$ by

$$\mathcal{U}_y = \{U \subset Y : \exists\, O \in \tau \text{ s.t. } y \in O \subset U\},$$

we call $\mathcal{B}_y \subset \mathcal{U}_y$ as *neighbourhood basis* of y in (Y,τ) if $\forall U \in \mathcal{U}_y\, \exists\, V \in \mathcal{B}_y$ s.t. $V \subset U$.

- (*Metrizability*) A topological space (Y,τ) is called *metrizable* if there exists a metric $d : Y \times Y \to \mathbb{R}$ on Y such that, denoting $B_\varepsilon(a) = \{y \in Y : d(y,a) < \varepsilon\}$ (for $a \in Y$, $\varepsilon \in (0,+\infty)$), there holds

$$\tau = \{O \subset Y \text{ s.t. } \forall\, a \in O\, \exists\, \varepsilon > 0 : B_\varepsilon(a) \subset O)\}.$$

(a) Show that any metrizable topology τ satisfies the first axiom of countability, which means that each point has a countable neighbourhood basis.

(b) Prove that

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}(-\varepsilon,\varepsilon) \text{ s.t. } n \in \mathbb{N},\, f_1, f_2, \ldots, f_n \in X^*,\, \varepsilon > 0 \right\}$$

is a neighbourhood basis of $0 \in X$ in (X,τ_{w}).

(c) Show that if (X, τ_{w}) is first countable, then $(X^*, \|\cdot|_{X^*})$ admits a countable algebraic basis.

Hint. Suppose (X, τ_{w}) is first countable and let $(U_j)_{j\in\mathbb{N}}$, be a countable neighborhood basis of 0. By part (b), for each U_j there exists an element

$$B_j = \bigcap_{k=1}^{n_j} f_{j,k}^{-1}((-\varepsilon_j, \varepsilon_j)) \in \mathcal{B}$$

contained in U_j, for some $n_j \in \mathbb{N}$, $f_{j,k} \in X^*$, $\varepsilon_j > 0$. Let now $\lambda \in X^*$, and consider the weakly open set $\lambda^{-1}((-1, 1))$. This contains B_j for some j. Show using Exercise 7.3-(a) that λ is a linear combination of $f_{j,k}$, $k = 1, \ldots, n_j$. Conclude that X^* admits a countable algebraic basis.

(d) Assume that X is infinite-dimensional and conclude from (a), (c) and Exercise 3.2-(b) that (X, τ_{w}) is not metrizable.

Solution.

(a) Let (Y, τ) be a metrizable topological space. Let $d : Y \times Y \to \mathbb{R}$ be a metric inducing the topology τ. Given $y \in Y$, we consider

$$B_\varepsilon(y) := \{z \in Y \mid d(y, z) < \varepsilon\} \text{ for } \varepsilon \in (0, \infty), \quad \mathcal{B}_y := \left\{B_{\frac{1}{n}}(y) \mid n \in \mathbb{N}\right\}.$$

Let U now be any neighbourhood of y. Since (Y, τ) is metrizable, there exists $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(y) \subseteq U$. Choosing $\mathbb{N} \ni n > \frac{1}{\varepsilon}$, we have $B_{\frac{1}{n}}(y) \subseteq U$, which shows that $\mathcal{B}_y$ is a neighbourhood basis of y in (Y, τ). Since $y \in Y^n$ is arbitrary and $\mathcal{B}_y$ countable, we have verified the first axiom of countability for (Y, τ).

(b) Let $U \subseteq X$ be any neighbourhood of $0 \in X$ in (X, τ_{w}). Then there exists $\Omega \in \tau_{\mathrm{w}}$ such that $0 \in \Omega \subseteq U$. By definition of weak topology, Ω is an arbitrary union and finite intersection of sets of the form $f^{-1}(I)$ for $f \in X^*$ and $I \subseteq \mathbb{R}$ open. In particular, Ω contains a finite intersection of such sets containing the origin. More precisely, there exist $f_1, \ldots, f_n \in X^*$ and open sets $I_1, \ldots, I_n \subseteq \mathbb{R}$ such that

$$\Omega \supseteq \bigcap_{k=1}^{n} f_k^{-1}(I_k) \ni 0.$$

By linearity $f_k(0) = 0 \in I_k$ for every $k \in \{1, \ldots, n\}$. Since $I_1, \ldots, I_n \subseteq \mathbb{R}$ are open and n finite, there exists $\varepsilon \in (0, \infty)$ such that $(-\varepsilon, \varepsilon) \subseteq I_k$ for every $k \in \{1, \ldots, n\}$. Thus,

$$\Omega \supseteq \bigcap_{k=1}^{n} f_k^{-1}((-\varepsilon, \varepsilon)) = \{x \in X | \forall k \in \{1, \ldots, n\} : |f_k(x)| < \varepsilon\}$$

and we conclude that a neighbourhood basis of $0 \in X$ in (X, τ_{w}) is given by

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}((-\varepsilon, \varepsilon)) \mid n \in \mathbb{N}, f_1, \ldots, f_n \in X^*, \varepsilon \in (0, \infty) \right\}.$$

(c) Let $(X, \|\cdot\|_X)$ be a normed space and suppose that (X, τ_{w}) is first countable. Then there exists a countable neighbourhood basis $\{A_\alpha\}_{\alpha \in \mathbb{N}}$ of $0 \in X$ in (X, τ_{w}). Since $\mathcal{B}$ defined in (b) is also a neighbourhood basis of $0 \in X$ in (X, τ_{w}), we have

$$\forall \alpha \in \mathbb{N} \quad \exists B_\alpha \in \mathcal{B} : \quad B_\alpha \subseteq A_\alpha.$$

By construction of $\mathcal{B}$, this means that

$$\forall \alpha \in \mathbb{N} \quad \exists n_\alpha \in \mathbb{N}, f_1^\alpha, \ldots, f_{n_\alpha}^\alpha \in X^*, \varepsilon_\alpha \in (0, \infty) : \\ B_\alpha := \{x \in X \mid \forall k \in \{1, \ldots, n_\alpha\} : \mid f_k^\alpha(x) \mid < \varepsilon_\alpha\} \subseteq A_\alpha.$$

In other words, the topology τ_{w} coincides with the topology $\mathcal{U}_F$ which is induced by $F = \bigcup_{\alpha \in \mathbb{N}} \bigcup_{k=1}^{n_\alpha} \{f_k^\alpha\}$ (cf. Exercise 7.3). According to Exercise 7.3-(b), $X^* \subseteq \operatorname{span}(F)$. In other words, F contains an algebraic basis of X^* and F is clearly countable.

(d) By (a) and (c), $(X^*, \|\cdot\|_{X^*})$ admits a countable algebraic basis. But since X is infinite-dimensional, $(X^*, \|\cdot\|_{X^*})$ is infinite-dimensional. Moreover, $(X^*, \|\cdot\|_{X^*})$ is a Banach space. But as such, according to Exercise 3.2-(b), it can only have a countable algebraic basis if it is finite-dimensional, a contradiction.

□